

Homework 3

Math 110

Due Thursday, June 30

Section 2.1: #10, 17, 21

- X1. Let V, W be two vector spaces, and $T \in L(V, W)$.
- (a) Suppose W' is a subspace of W , and define $T^{-1}W' = \{x \in V : Tx \in W'\}$. Show that $T^{-1}W'$ is a subspace of V .
 - (b) Suppose V' is a subspace of V , and define $TV' = \{Tx : x \in V'\}$. Show that TV' is a subspace of W .
 - (c) Identify the subspaces $T^{-1}(\{0\})$ and $T^{-1}(W)$ of V , and the subspaces $T(\{0\})$ and $T(V)$ of W .
- X2. Let V, W be two vector spaces, and $T \in L(V, W)$. Suppose we have a basis $\{x_1, x_2, \dots, x_n\}$ of V .
- (a) Prove that T is one-to-one if and only if Tx_1, Tx_2, \dots, Tx_n are distinct, and $\{Tx_1, Tx_2, \dots, Tx_n\}$ is linearly independent.
 - (b) Prove that T is onto if and only if $\{Tx_1, Tx_2, \dots, Tx_n\}$ spans W .
- X3. Let W_1, W_2 be finite-dimensional subspaces of V , and define $\alpha : W_1 \oplus W_2 \rightarrow V$ by $\alpha(x, y) = x + y$ for $x \in W_1, y \in W_2$.
- (a) Show that α is linear.
 - (b) Prove that $\text{im } \alpha = W_1 + W_2$.
 - (c) Prove that $\ker \alpha \simeq W_1 \cap W_2$.
 - (d) Use this to conclude that $\dim W_1 + \dim W_2 = \dim(W_1 + W_2) + \dim(W_1 \cap W_2)$.
 - (e) Prove that α is an isomorphism if and only if $W_1 \cap W_2 = \{0\}$ and $W_1 + W_2 = V$. This explains the notation $V \simeq W_1 \oplus W_2$ for this situation.
- X4. Let V, W be \mathbb{Q} -vector spaces, and let $T : V \rightarrow W$ be a function which preserves addition (i.e. $T(x + y) = Tx + Ty$ whenever $x, y \in V$). Prove that T is automatically \mathbb{Q} -linear. *Hint:* First prove that T preserves scalar multiples by nonnegative integers.

Section 2.2: #4, 9, 13

Section 2.4: #3