Homework 2 Solutions

Math 110

Section 1.4

13. We have that $\text{span}(S_2)$ is a subspace of V which contains S_1; therefore, $\text{span}(S_2)$ must contain all of $\text{span}(S_1)$.

    Alternately, any linear combination of vectors in S_1 is also a linear combination of vectors in S_2.

14. Since $S_1 \subseteq \text{span}(S_1)$ and $S_2 \subseteq \text{span}(S_2)$, we have $S_1 \cup S_2 \subseteq \text{span}(S_1) + \text{span}(S_2)$; therefore, since $\text{span}(S_1) + \text{span}(S_2)$ is a subspace of V which contains $S_1 \cup S_2$, we get $\text{span}(S_1 \cup S_2) \subseteq \text{span}(S_1) + \text{span}(S_2)$. On the other hand, by the previous problem, both $\text{span}(S_1)$ and $\text{span}(S_2)$ are contained in the subspace $\text{span}(S_1 \cup S_2)$; therefore, by problem 1.3.23, $\text{span}(S_1) + \text{span}(S_2) \subseteq \text{span}(S_1 \cup S_2)$ also. Combining the two inclusions, we must have $\text{span}(S_1 \cup S_2) = \text{span}(S_1) + \text{span}(S_2)$.

    Alternately, a linear combination of vectors in S_1 plus a linear combination of vectors in S_2 together give a linear combination of vectors in $S_1 \cup S_2$; on the other hand, given a linear combination of vectors in $S_1 \cup S_2$, we can split the sum into a part with vectors in S_1 and a part with vectors in S_2.

Section 1.5

8. (a) Suppose we have $a_1(1,1,0) + a_2(1,0,1) + a_3(0,1,1) = 0$; then $(a_1 + a_2, a_1 + a_3, a_2 + a_3) = 0$; in other words,

    $$\begin{aligned} a_1 + a_2 &= 0, \\ a_1 + a_3 &= 0, \\ a_2 + a_3 &= 0. \end{aligned}$$

    However, adding the first two equations gives $2a_1 + a_2 + a_3 = 0$; now subtracting the third equation gives $2a_1 = 0$, so $a_1 = 0$. From the above equations, this implies $a_2 = 0$ and $a_3 = 0$ also.

    (b) If F has characteristic 2, then $1(1,1,0) + 1(1,0,1) + 1(0,1,1) = (2,2,2) = (0,0,0)$, showing that the three vectors are linearly dependent.

10. Consider the three vectors $(1,0,0)$, $(0,1,0)$, $(1,1,0)$. Then none of the three is a scalar multiple of another, but $1(1,0,0) + 1(0,1,0) + (-1)(1,1,0) = (0,0,0)$, so the three vectors are linearly dependent.

15. ($\Rightarrow$) Suppose S is linearly dependent; then find $a_1, \ldots, a_n$ not all zero such that $a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0$. Let m be the largest number such that $a_m \neq 0$; then we have $a_1u_1 + a_2u_2 + \cdots + a_mu_m = 0$. Solving for u_m, we have

    $$u_m = -\frac{a_1}{a_m}u_1 - \frac{a_2}{a_m}u_2 - \cdots - \frac{a_{m-1}}{a_m}u_{m-1}.$$

    Thus $u_m \in \text{span}\{u_1, u_2, \ldots, u_{m-1}\}$. (If $m = 1$, this means $u_1 = 0$.)

    ($\Leftarrow$) If $u_1 = 0$, then S is linearly dependent since any set with the zero vector is linearly dependent. If $u_{k+1} \in \text{span}\{u_1, u_2, \ldots, u_k\}$, then u_{k+1} is a linear combination of the other vectors in S, so S is linearly dependent in this case also.

20. Suppose we have constants a, b such that $af + bg = 0$; then $ae^{rt} + be^{st} = 0$ for all $t \in \mathbb{R}$. In particular, plugging in $t = 0$, we get $a + b = 0$; similarly, plugging in $t = 1$, we get $ae^r + be^s = 0$. Therefore, $b = -a$, so $a(e^r - e^s) = 0$; however, since $r \neq s$, we have $e^r \neq e^s$, so $e^r - e^s \neq 0$. Therefore, we must have $a = 0$, and $b = -a = 0$.

Section 1.6

11. We have $u = \frac{1}{a}(au)$ and $v = (u + v) - \frac{1}{a}(au)$ are both in the span of $\{u + v, au\}$; therefore, since $\{u, v\}$ spans V, so does $\{u + v, au\}$. Since this set has size equal to the dimension of V, it must therefore be a basis for V.

    Similarly, $u = \frac{1}{a}(au)$ and $v = \frac{1}{b}(bv)$ are both in the span of $\{au, bv\}$, so $\{au, bv\}$ spans V. Again, since this set has size equal to that of a known basis, $\{au, bv\}$ must be a basis for V.

13. Subtracting 2 times the first equation from the second gives $x_2 - x_3 = 0$; now adding 2 times this equation to the first equation gives $x_1 - x_3 = 0$. Therefore, the general solution to the system of equations is $(x_1, x_2, x_3) = (t, t, t)$ for $t \in \mathbb{R}$. Since this is equal to $t(1, 1, 1)$, this subspace is generated by $\{(1, 1, 1)\}$, which is also obviously linearly independent. Therefore, $\{(1, 1, 1)\}$ is a basis for the solution space.

33. (a) First, any vector in $\beta_1 \cap \beta_2$ must be in $W_1 \cap W_2 = \{0\}$; however, since the zero vector cannot be in any basis, this implies that $\beta_1 \cap \beta_2 = \emptyset$. Now to see $\beta_1 \cup \beta_2$ is a basis, first by problem 1.4.14, the span of $\beta_1 \cup \beta_2$ is $W_1 + W_2 = V$. Now suppose $a_1x_1 + \cdots + a_nx_n + b_1y_1 + \cdots + b_my_m = 0$ with $x_1, \ldots, x_n \in \beta_1$, $y_1, \ldots, y_m \in \beta_2$ all distinct. Then $a_1x_1 + \cdots + a_nx_n = -b_1y_1 - \cdots - b_my_m$; from the left hand side, this vector is in W_1, and from the right hand side, it is in W_2. Since $W_1 \cap W_2 = \{0\}$, both sides must be zero. However, since β_1 and β_2 are linearly independent, this implies $a_1 = \cdots = a_n = 0$ and $-b_1 = \cdots = -b_m = 0$, so $b_1 = \cdots = b_m = 0$ also.

    (b) First, we have $V = \text{span}(\beta_1 \cup \beta_2) = \text{span}(\beta_1) + \text{span}(\beta_2) = W_1 + W_2$. Now to see $W_1 \cap W_2 = \{0\}$, suppose $x \in W_1 \cap W_2$. Since $x \in W_1$, we may write $x = a_1x_1 + \cdots + a_nx_n$ with $x_1, \ldots, x_n \in \beta_1$ distinct, and since $x \in W_2$, we may write $x = b_1y_1 + \cdots + b_my_m$ with $y_1, \ldots, y_m \in \beta_2$ distinct. We now have

    $$a_1x_1 + \cdots + a_nx_n - b_1y_1 - \cdots - b_my_m = 0.$$

    Since β_1 and β_2 are disjoint, all the vectors $x_1, \ldots, x_n, y_1, \ldots, y_m$ are distinct. However, $\beta_1 \cup \beta_2$ is linearly independent, so $a_1 = \cdots = a_n = -b_1 = \cdots = -b_m = 0$. In particular, we have $x = a_1x_1 + \cdots + a_nx_n = 0$.

34. (a) Choose a basis $\{x_1, \ldots, x_n\}$ of W_1, and extend this to a basis $\{x_1, \ldots, x_n, y_1, \ldots, y_m\}$ of V. Then if we let $W_2 = \text{span}\{y_1, \ldots, y_m\}$, then $V = W_1 \oplus W_2$ by the previous problem.

    (b) If we let $W_2 = \{(0, a_2) : a_2 \in \mathbb{R}\}$ and $W_2' = \{(t, t) : t \in \mathbb{R}\}$, then $V = W_1 \oplus W_2$ and $V = W_1 \oplus W_2'$. This is because $\beta_1 = \{(1, 0)\}$ is a basis for W_1; $\beta_2 = \{(0, 1)\}$ is a basis for W_2; and $\beta_2' = \{(1, 1)\}$ is a basis for W_2'. Now $\beta_1 \cap \beta_2 = \beta_1 \cap \beta_2' = \emptyset$, and $\beta_1 \cup \beta_2$ and $\beta_1 \cup \beta_2'$ are both linearly independent subsets of $\mathbb{R}^2$ of size 2 and therefore bases.

X1. (a) Since $\dim(W_1 \cap W_2) = \dim W_1 + \dim W_2 - \dim(W_1 + W_2)$, and $\dim(W_1 + W_2) \leq \dim \mathbb{R}^3 = 3$, we get

    $$\dim(W_1 \cap W_2) \geq 2 + 2 - 3 = 1.$$

(b) Suppose $x \in W_1 \cap W_2$; then since $x \in W_1$, we can write $x = a_1(1,2,3) + a_2(0,1,1)$, and since $x \in W_2$, we can write $x = b_1(2,3,2) + b_2(0,1,0)$. Subtracting, this implies that $a_1(1,2,3) + a_2(0,1,1) - b_1(2,3,2) - b_2(0,1,0) = 0$; in other words,

$$\begin{aligned} a_1 - 2b_1 &= 0, \\ 2a_1 + a_2 - 3b_1 - b_2 &= 0, \\ 3a_1 + a_2 - 2b_1 &= 0. \end{aligned}$$

Solving this system of equations by the methods of math 54, we see the general solution is $(a_1, a_2, b_1, b_2) = t(-\frac{2}{3}, \frac{4}{3}, -\frac{1}{3}, 1)$. Now going back to x, we see that $x = -\frac{2}{3}t(1,2,3) + \frac{4}{3}t(0,1,1) = t(-\frac{2}{3}, 0, -\frac{2}{3}) = -\frac{1}{3}t(2,3,2) + t(0,1,0)$. Therefore, $\{(-\frac{2}{3}, 0, -\frac{2}{3})\}$ is a basis for $W_1 \cap W_2$; multiplying by $-\frac{3}{2}$, we get the prettier basis $\{(1,0,1)\}$ for $W_1 \cap W_2$.

Alternately, one can prove that $W_1 = \{(x,y,z) : z = x + y\}$ and $W_2 = \{(x,y,z) : z = x\}$. Therefore, if $(x,y,z) \in W_1 \cap W_2$, then $z = x + y = x$, so $y = 0$, and $(x,y,z) = x(1,0,1)$.

X2. Let $\{x_1, \dots, x_n\}$ be a basis for V as a $\mathbb{C}$-vector space; this means that every vector of V can be uniquely written as $a_1x_1 + a_2x_2 + \cdots + a_nx_n$ for some $a_1, \dots, a_n \in \mathbb{C}$. Writing each a_j as $b_j + c_ji$ for $b_j, c_j \in \mathbb{R}$, this implies that every vector of V can be written in the form

$$b_1x_1 + c_1(ix_1) + b_2x_2 + c_2(ix_2) + \cdots + b_nx_n + c_n(ix_n),$$

with real scalars $b_1, c_1, b_2, c_2, \dots, b_n, c_n$; in other words, $\{x_1, ix_1, x_2, ix_2, \dots, x_n, ix_n\}$ spans V as an $\mathbb{R}$-vector space. On the other hand, if such a linear combination is zero, then

$$(b_1 + ic_1)x_1 + (b_2 + ic_2)x_2 + \cdots + (b_n + ic_n)x_n = 0.$$

Since $\{x_1, \dots, x_n\}$ are linearly independent in V as a $\mathbb{C}$-vector space, this implies $b_1 + ic_1 = b_2 + ic_2 = \cdots = b_n + ic_n = 0$, so $b_1 = c_1 = b_2 = c_2 = \cdots = b_n = c_n = 0$. Thus, $\{x_1, ix_1, x_2, ix_2, \dots, x_n, ix_n\}$ is linearly independent in V as an $\mathbb{R}$-vector space. Therefore, $\{x_1, ix_1, x_2, ix_2, \dots, x_n, ix_n\}$ forms a basis for V as an $\mathbb{R}$-vector space. Since this basis has size $2n$, this shows that $\dim_{\mathbb{R}} V = 2n = 2\dim_{\mathbb{C}} V$.

X3. Let $\{x_1, \dots, x_m\}$ be a basis for V, and let $\{y_1, \dots, y_n\}$ be a basis for W. We then claim that

$$\beta = \{(x_1, 0), \dots, (x_m, 0), (0, y_1), \dots, (0, y_n)\}$$

is a basis for $V \oplus W$. To see this, first suppose we have $(x, y) \in V \oplus W$, with $x \in V$ and $y \in W$. Then if we write $x = a_1x_1 + \cdots + a_mx_m$ and $y = b_1y_1 + \cdots + b_ny_n$, then

$$\begin{aligned} &a_1(x_1, 0) + \cdots + a_m(x_m, 0) + b_1(0, y_1) + \cdots + b_n(0, y_n) \\ &\qquad = (a_1x_1 + \cdots + a_mx_m, b_1y_1 + \cdots + b_ny_n) = (x, y). \end{aligned}$$

Therefore, β spans $V \oplus W$. On the other hand, suppose

$$\begin{aligned} &a_1(x_1, 0) + \cdots + a_m(x_m, 0) + b_1(0, y_1) + \cdots + b_n(0, y_n) \\ &\qquad = (a_1x_1 + \cdots + a_mx_m, b_1y_1 + \cdots + b_ny_n) = (0, 0). \end{aligned}$$

Then $a_1x_1 + \cdots + a_mx_m = 0$; since $\{x_1, \dots, x_m\}$ is linearly independent, this implies $a_1 = \cdots = a_m = 0$. Similarly, $b_1y_1 + \cdots + b_ny_n = 0$; since $\{y_1, \dots, y_n\}$ is linearly independent, this implies $b_1 = \cdots = b_n = 0$ also. Therefore, β is linearly independent.

We now have $\dim(V \oplus W) = |\beta| = m + n = \dim V + \dim W$.

X4. We first show that the given set spans V; thus, suppose $x \in V$. Then since $\{x_1, \ldots, x_n\}$ spans V/W, there are scalars $a_1, \ldots, a_n$ such that $x + W = a_1(x_1 + W) + \cdots + a_n(x_n + W) = (a_1x_1 + \cdots + a_nx_n) + W$. This means that $x - a_1x_1 - \cdots - a_nx_n \in W$; since $\{y_1, \ldots, y_m\}$ spans W, this implies that there are scalars $b_1, \ldots, b_m$ such that

$$x - a_1x_1 - \cdots - a_nx_n = b_1y_1 + \cdots + b_my_m.$$

Therefore, $x = a_1x_1 + \cdots + a_nx_n + b_1y_1 + \cdots + b_my_m$.

Now to see the given set is linearly independent, suppose $a_1x_1 + \cdots + a_nx_n + b_1y_1 + \cdots + b_my_m = 0$. Then we have

$$a_1(x_1 + W) + \cdots + a_n(x_n + W) = (a_1x_1 + \cdots + a_nx_n) + W = 0 + W,$$

since $a_1x_1 + \cdots + a_nx_n = -b_1y_1 - \cdots - b_my_m \in W$. Since $\{x_1 + W, \ldots, x_n + W\}$ is linearly independent, this implies $a_1 = \cdots = a_n = 0$. Plugging in, we get $b_1y_1 + \cdots + b_my_m = 0$; since $\{y_1, \ldots, y_m\}$ is linearly independent, this implies $b_1 = \cdots = b_m = 0$ also.

X5. (a) Choose a basis $\{x_1, \ldots, x_n\}$ of V; then each vector in V can be written uniquely as a linear combination $a_1x_1 + \cdots + a_nx_n$. Since we have p choices for each scalar a_i, and the choices are independent, V has p^n elements.

(b) Each subspace of dimension 1 has a basis which consists of a single nonzero vector in V. However, this overcounts, since each such subspace has $p - 1$ nonzero vectors, and thus $p - 1$ bases. Therefore, there are $\frac{p^n-1}{p-1}$ subspaces of dimension 1.

(c) First, to choose a nonzero x, we have $p^n - 1$ choices. Now we need to choose y which is not a scalar multiple of x; since there are p such scalar multiples, we have $p^n - p$ choices. Therefore, there are $(p^n - 1)(p^n - p)$ choices of the ordered pair x, y.

(d) Each pair x, y from the previous part spans a subspace of dimension 2. However, since each subspace has multiple bases, this overcounts; in particular, each subspace of dimension 2 has $(p^2 - 1)(p^2 - p)$ ordered bases. (This is a special case of the previous part, since an ordered basis of a subspace W of dimension 2 is equivalent to an ordered pair of vectors $x, y \in W$ which are linearly independent.) Therefore, the total number of subspaces of dimension 2 is

$$\frac{(p^n - 1)(p^n - p)}{(p^2 - 1)(p^2 - p)} = \frac{(p^n - 1)(p^{n-1} - 1)}{(p^2 - 1)(p - 1)}.$$

Section 1.7

3. We claim that $\{1, \pi, \pi^2, \pi^3, \cdots\}$ is an infinite linearly independent subset of $\mathbb{R}$, which implies $\dim_{\mathbb{Q}} \mathbb{R}$ is infinite. To see this, suppose some finite linear combination

$$a_0 \cdot 1 + a_1\pi + a_2\pi^2 + \cdots + a_n\pi^n = 0,$$

with $a_0, \ldots, a_n \in \mathbb{Q}$. Then $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ is a polynomial with π as a zero; since π is transcendental, this implies the polynomial must be the zero polynomial, so $a_n = a_{n-1} = \cdots = a_0 = 0$.

Alternately, one can prove that any finite-dimensional vector space over $\mathbb{Q}$ is countable; since $\mathbb{R}$ is uncountable, this proves the desired result.