A local André-Oort theorem via ACFA

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An Introduction to Recent Applications of Model Theory
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Special varieties from special points

Definition

Let X be a variety over the field K and $\Xi \subseteq X(K)$ a set of K-rational points on X. We say that $Y \subseteq X^n$, a subvariety of a Cartesian power of X, is Ξ-special just in case $Y(K) \cap \Xi^n$ is Zariski dense in Y.

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Problem

Given a variety X and a set $\Xi \subseteq X(K)$ describe the special varieties.

Theorems on special points: Manin-Mumford conjecture

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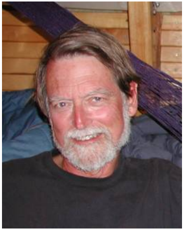

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- Ξ *is the forward orbit of some specified point?*

André-Oort conjecture, first form

Conjecture (André-Oort)

Let S be a Shimura variety and $\Xi \subseteq S(\mathbb{C})$ the set of special points on S. Then a subvariety $X \subseteq S$ is Ξ-special if and only if it is special

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In general, if S is a Shimura variety, then $S(\mathbb{C})$ admits an analytic uniformization as $\Gamma \backslash G_0 / K$ where

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The special subvarieties are subvarieties which contain at least one special point and which are naturally uniformized by homogeneous spaces.

Moduli spaces as Shimura varieties

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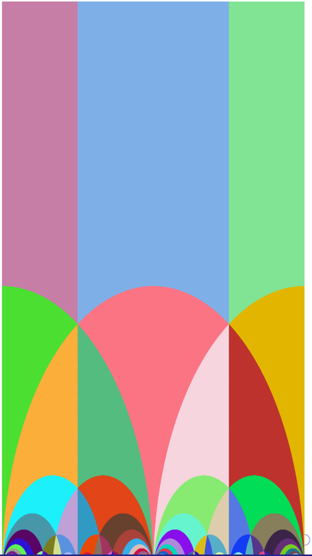
Informally, a moduli space for abelian varieties is an algebraic variety $\mathscr{A}$ for which the points of $\mathscr{A}$ correspond in a natural way to isomorphism classes of abelian varieties with some extra structure. The special points in $\mathscr{A}(\mathbb{C})$ are the points corresponding to those abelian varieties having complicated endomorphism rings.

Analytic presentation of modular curves

Every elliptic curve (over $\mathbb{C}$) may be realized as a complex torus of the form $E_\tau := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ for some $\tau \in \mathfrak{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$.
$PSL_2(\mathbb{R})$ acts transitively and faithfully on $\mathfrak{H}$ via fractional linear transformations and $E_\tau \cong E_{\tau'}$ just in case $\tau' = \gamma(\tau)$ for some $\gamma \in PSL_2(\mathbb{Z})$.
Thus, we may identify the moduli space of elliptic curves with $Y_0(1) := \mathbb{A}^1 \xleftarrow{j} PSL_2(\mathbb{Z})\backslash\mathfrak{H} = PSL_2(\mathbb{Z})\backslash PSL_2(\mathbb{R})/K$ where K is the stabilzer of $i = \sqrt{-1}$.



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- A modular curve $X \subseteq \mathbb{A}^2$ is an algebraic curve parametrizing the sets of pairs of elliptic curves (E_1, E_2) for which there exists an isogeny $f : E_1 \to E_2$ of some specified kind (e.g. having a cyclic kernel of size 38).

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- The class of special subvarieties of $\mathbb{A}^n$ is generated from the special points and the modular curves by taking fibres and intersections.

André-Oort theorems for modular curves

Theorem (André, Edixhoven, Yafaev)

If $X \subseteq \mathbb{A}^n$ is a curve containing a dense set of CM-points, then X is a special.

Universal abelian varieties

Example

- The modular curve $Y_1(N)$ parametrizes isomorphisms classes of pairs (E, α) where E is an elliptic curve and $\alpha : (\mathbb{Z}/N\mathbb{Z})^2 \to E[n]$ is an isomorphism identifying the N-torsion group with $(\mathbb{Z}/N\mathbb{Z})^2$. If $N \geq 3$, then there is a universal elliptic curve $\pi : \mathscr{E} \to Y_1(N)$ (along with subvarieties $\Psi_{(0,0)}, \ldots, \Psi_{(N-1,N-1)} \subseteq \mathscr{E}$) so that any point $a \in Y_1(N)(R)$ is the moduli point of $(\mathscr{E}_a, \alpha_a)$ where $\alpha_a(i,j) := (\Psi_{(i,j)})_a$.

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- More generally, there are universal abelian varieties $\pi : \mathscr{X} \to \mathscr{A}$ over moduli spaces $\mathscr{A}$ for abelian varieties of dimension g with fixed polarization and level type, provided that the level is big enough.

André-Oort + Manin-Mumford + Mordell-Lang

Let $\pi : \mathscr{X} \to \mathscr{A}$ be a universal abelian variety over a moduli space. A point $\zeta \in \mathscr{X}(\mathbb{C})$ is special if $X := \mathscr{X}_\zeta$ is a CM-abelian variety and ζ is a torsion point in $X(\mathbb{C})$. A subvariety $Y \subseteq \mathscr{X}$ is special if $\pi(Y) \subseteq \mathscr{A}$ is a special subvariety (or variety of Hodge type) in the old sense, and $[N](Y)$ is a group scheme over $\pi(Y)$ for some $N \in \mathbb{Z}_+$.

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Conjecture

An irreducible subvariety $Y \subset \mathscr{X}$ contains a dense set of special points if and only if it is special.

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This conjecture has been formulated in more general terms by André for mixed Shimura varieties and by Pink in terms of generalized Hecke orbits so as to include the Mordell-Lang conjecture as well.

ACFA and Manin-Mumford

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Hrushovski reproved this theorem and in the process produced effective bounds in it statement by applying the model theoretic analysis of difference fields.

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Theorem (Chatzidakis, Hrushovski, Peterzil)

The theory of difference fields has a supersimple model companion ACFA for which the minimal types satisfy the Zilber trichotomy in the sense that a minimal type which is not one-based is non-orthogonal to a definable field.

Difference field strategy for Manin-Mumford

In the case of the Manin-Mumford conjecture, we are given an abelian variety A over $\mathbb{C}$. Look for an automorphism $\sigma : \mathbb{C} \to \mathbb{C}$ and a subgroup $\Gamma < A(\mathbb{C})$ definable in $\mathscr{L}(+, \times, \sigma, \{c\}_{c \in \mathbb{C}})$ for which

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The theorem follows using the structure of one-based groups.

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Finding a good difference equation for all of the special points is far trickier in this case.

Canonical lifts as p-adic special points

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- Conversely, every CM abelian variety is a canonical lift at infinitely many primes.

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Theorem (Moonen)

Let $\mathscr{A}$ be a moduli space of abelian varieties over R. If $Y \subseteq \mathscr{A}$ is an irreducible subvariety containing a Zariski dense set of ordinary canonical moduli points, then Y is special.

Here an ordinary canonical moduli point is a point in $\mathscr{A}(R)$ encoding an abelian scheme A over R which is a canonical lift and whose reduction A_k is ordinary, meaning that $\#(A_k(k)[p]) = p^{\dim A}$.

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Theorem (Scanlon)

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- From this, one concludes that if there is a counterexample, one can be found where $\pi : (Y(K) \cap \widehat{\Upsilon}) \to (\pi(Y)(K) \cap \Upsilon)$ is generically finite.

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- By Moonen's theorem, $\pi(Y)$ is a special variety. It follows from this, the fact that $\pi : Y \to \pi(Y)$ is finite, and the fact that Y contains a dense set of torsion points on ordinary canonical fibres that Y is actually defined over a number field and contains a dense set of torsion points on ordinary canonical fibres for infinitely many primes ℓ.

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- Using a theorem of Edixhoven and Yafaev, it is easy to see that this is enough.

Model theoretic approach to $\mathscr{A}$

Question

Can one recover Moonen's theorem via the model theory of valued difference fields?

Model theoretic approach to $\mathscr{A}$

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Uniform versions of Moonen's theorem can be extracted model theoretically as a consequence of its truth.

Model theoretic approach to $\mathscr{A}$

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Can one recover Moonen's theorem via the model theory of valued difference fields?

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Can the ACFA approach give a trick for mixing information at different primes as in Hrushovski's proof of Manin-Mumford?

Drinfeld modular varieties

Theorem (Breuer)

The natural analogue of the André-Oort conjecture for the moduli space of rank two Drinfeld modules is true.

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Question

Can one deduce the analogous André-Oort + Manin-Mumford conjecture for the universal Drinfeld module?

The End