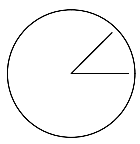
Chapter 8: Trigonometric Functions

A circle of radius 1 has a circumference of
$2\pi \approx 6.283185307179586476925286766559005768394338798750211$.

When measuring angles by degrees ($^\circ$), a full revolution is 360°.
For our trigonometric functions, we use *radians* as our arguments.

To convert between degrees and radians, one should find the arc length of the segment of the unit circle demarked by two radii meeting at an angle of x°.

Conversion from degrees to radians



From the equality

$$360^\circ = 2\pi$$

we find that x° corresponds to $\frac{\pi}{180}x$ radians.

Imporant conversions

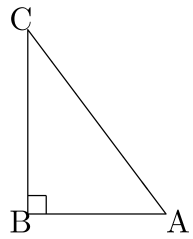
$$\begin{aligned} 30^\circ &\Leftrightarrow \frac{\pi}{6} \\ 45^\circ &\Leftrightarrow \frac{\pi}{4} \\ 60^\circ &\Leftrightarrow \frac{\pi}{3} \\ 90^\circ &\Leftrightarrow \frac{\pi}{2} \\ 180^\circ &\Leftrightarrow \pi \\ 360^\circ &\Leftrightarrow 2\pi \end{aligned}$$

Negative arc lengths

A negative arc length should be interpreted as a distance along the unit circle in the *clockwise* direction.

Sine and Cosine

In terms of triangles, the sine of an angle $\angle A$ is the ratio of the length of the opposite side by the length of the hypotenuse. The cosine of the angle is the ratio of the length of the adjacent side by the length of the hypotenuse.

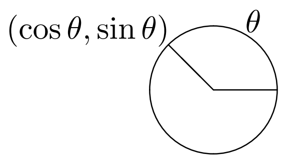


$$\sin(\angle A) = \frac{\overline{BC}}{\overline{CA}}$$

$$\cos(\angle A) = \frac{\overline{AB}}{\overline{CA}}$$

Sine and Cosine via circles

The point on the unit circle θ radians counterclockwise from $(1, 0)$ is $(\cos\theta, \sin\theta)$.



Some values

$$\sin(0) = 0$$

$$\cos(0) = 1$$

$$\sin(\frac{\pi}{4}) = 1 = \cos(\frac{\pi}{4})$$

$$\sin(\frac{\pi}{6}) = \frac{1}{2}$$

$$\cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$$

$$\sin(\frac{\pi}{2}) = 1$$

$$\cos(\pi) = -1$$

Trigonometric identities

If n is a positive integer, then one write $\sin^n(x)$ for $(\sin(x))^n$. Likewise, for the cosine and other trigonometric functions.

The Pythagorean theorem may be expressed as:

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

As an arclength of 2π corresponds to a full revolution of the circle,

$$\sin(\theta + 2\pi) = \sin(\theta)$$

$$\cos(\theta + 2\pi) = \cos(\theta)$$

More identities

$$\sin(-\theta) = -\sin(\theta)$$
$$\cos(-\theta) = \cos(\theta)$$

$$\sin(\frac{\pi}{2} - \theta) = \cos(\theta)$$

$$\sin(\theta + \varphi) = \cos(\theta)\sin(\varphi) + \sin(\theta)\cos(\varphi)$$

$$\cos(\theta + \varphi) = \cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi)$$

Differentiation of Sine and Cosine

$$\frac{d}{dx}\sin(x) = \cos(x)$$
$$\frac{d}{dx}\cos(x) = -\sin(x)$$

Indefinite integration of Sine and Cosine

$$\int \sin(x)dx = -\cos(x) + C$$

$$\int \cos(x)dx = \sin(x) + C$$

Computing derivatives involving trigonometric functions

Let $g(x) = \sin(x^2 + \cos(e^x))$. Compute $g'(x)$.

Solution

$$
\begin{aligned}
g'(x) &= \frac{d}{dx}(\sin(x^2 + \cos(e^x))) \\
&= \cos(x^2 + \cos(e^x))\frac{d}{dx}(x^2 + \cos(e^x)) \\
&= \cos(x^2 + \cos(e^x))(2x - \sin(e^x)e^x)
\end{aligned}
$$