General method

If $R := \{(x,y) \mid a \leq x \leq b, g(x) \leq y \leq h(x)\}$ then

$$\int\int_R f(x,y)dxdy = \int_a^b (\int_{g(x)}^{h(x)} f(x,y)dy)dx$$

To compute the iterated integral, first find $F(x,y)$ for which $\frac{\partial F}{\partial y} = f(x,y)$ so that

$$\int_{g(x)}^{h(x)} f(x,y)dy = F(x,y)|_{y=g(x)}^{y=h(x)} = F(x,h(x)) - F(x,g(x))$$

Then find $\Phi(x)$ for which $\Phi'(x) = F(x,h(x)) - F(x,g(x))$ and we find

$$\int_a^b (F(x,h(x)) - F(x,g(x)))dx = \Phi(b) - \Phi(a)$$

Putting these together, we have

$$\int\int_R f(x,y)dxdy = \Phi(b) - \Phi(a)$$

Example

Let $R = \{(x, y) \mid 1 \le x \le 2, 1 \le y \le e^x\}$. Compute

$$\int\int_R \frac{x}{y} dx dy$$

Solution

$$\begin{aligned}
\int\int_R \frac{x}{y} dx dy &= \int_1^2 (\int_1^{e^x}) \frac{x}{y} dy) dx \\
&= \int_1^2 (x \ln(y)|_{y=1}^{y=e^x}) dx \\
&= \int_1^2 (x \ln(e^x) - x \ln(1)) dx \\
&= \int_1^2 x^2 dx \\
&= \frac{1}{3} x^3 |_{x=1}^{x=2} \\
&= \frac{8}{3} - \frac{1}{3} \\
&= \frac{7}{3}
\end{aligned}$$

Another example

Let $R = \{(x,y) \mid 1 \le x \le 3, \ln(x) \le y \le 4\}$. Compute

$$\int\int_R xe^y dxdy$$

Solution

$$\begin{aligned}
\int\int_R e^{x+y} dxdy &= \int_1^3 (\int_{\ln(x)}^4 xe^y dy) dx \\
&= \int_1^3 (xe^y|_{y=\ln(x)}^{y=4}) dx \\
&= \int_1^3 (x^2 - e^4 x) dx \\
&= (\frac{1}{3}x^3 - \frac{e^4}{2}x^2)|_{x=1}^{x=3} \\
&= 9 - \frac{9}{2}e^4 - \frac{1}{3} + e^4\frac{1}{2} \\
&= \frac{26}{3} - 4e^4
\end{aligned}$$

Changing variables

If the region R is defined by a constant bound on y with the bound on x varying as a function of y, then one may compute a double integral over R as an iterated integral where one first integrates with respect to x and then with respect to y.

Example

Let $R = \{(x, y) \mid y \leq x \leq 2y, 1 \leq y \leq 2\}$. Compute

$$\int\int_R e^{x+y} dx dy$$

$$
\begin{aligned}
\int\int_R e^{x+y}dxdy &= \int_1^2 (\int_y^{2y} e^{x+y}dx)dy \\
&= \int_1^2 (\int_y^{2y} e^y e^x dx)dy \\
&= \int_1^2 (e^y e^x|_{x=y}^{x=2y})dy \\
&= \int_1^2 (e^y e^{2y} - e^y e^y)dy \\
&= \int_1^2 (e^{3y} - e^{2y})dy \\
&= (\frac{1}{3}e^{3y} - \frac{1}{2}e^{2y})|_{y=1}^{y=3} \\
&= \frac{1}{3}e^9 - \frac{1}{2}e^6 - \frac{1}{3}e^3 + \frac{1}{2}e^2
\end{aligned}
$$

Rectangles and Fubini's Theorem

If the region R is a rectangle with horizontal and vertical sides relative to the coordinate axes, the one may compute a double integral over R as an iterated integral by first integrating with respect to x and then with respect to y or *vice versa*.

Example

Let $R = \{(x, y) \mid 0 \le x \le 2, 2 \le y \le 5\}$. Compute

$$\int\int_R \frac{x^2}{y} dxdy$$

Solution

$$\begin{aligned}
\int\int_R \frac{x^2}{y} dxdy &= \int_0^2 (\int_2^5 \frac{x^2}{y} dy) dx \\
&= \int_0^2 (x^2 \ln(y)|_{y=2}^{y=5}) dx \\
&= \int_0^2 [\ln(5)x^2 - \ln(2)x^2] dx \\
&= \int_0^2 [\ln(\frac{5}{2})x^2] dx \\
&= [\ln(\frac{5}{2})\frac{1}{3}x^3]|_{x=0}^2 \\
&= \frac{8}{3}\ln(\frac{5}{2})
\end{aligned}$$

Another solution

$$
\begin{aligned}
\int\int_R \frac{x^2}{y}dxdy &= \int_2^5 (\int_0^2 \frac{x^2}{y}dx)dy \\
&= \int_2^5 [\frac{1}{3}\frac{x^3}{y}|_{x=0}^{x=2}]dy \\
&= \int_2^5 [\frac{8}{3y}]dy \\
&= \frac{8}{3}\ln(y)|_{y=2}^{y=5} \\
&= \frac{8}{3}(\ln(5) - \ln(2)) \\
&= \frac{8}{3}\ln(\frac{5}{2})
\end{aligned}
$$