Section 7.7: Double integrals

If R is some region in the plane and $f(x, y)$ is a (continuous) function taking positive values, then the volume of $\{(x, y, z) \mid (x, y) \text{ in } R \text{ and } 0 \leq z \leq f(x, y)\}$ is the integral

$$\int\int_R f(x, y) dx\, dy$$

More generally, if $f(x, y)$ is any continuous function, then $\int\int_R f(x, y) dx\, dy$ is the *signed volume* of the solid bounded by f.

Example

If $R := \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$ is the unit square and $f(x, y) \equiv 1$, then the solid bounded by f over R is the unit cube. So, $\int\int_R f(x, y) dx\, dy = 1$.

Iterated integrals

If $R = \{(x,y) \mid a \le x \le b, g(x) \le y \le h(x)\}$ (where $a \le b$ are constants and g and h are continuous functions of x), then

$$\int\int_R f(x,y)dx\,dy = \int_a^b (\int_{g(x)}^{h(x)} f(x,y)dy)dx$$

Here, $F(x) = \int_{g(x)}^{h(x)} f(x,y)dy$ is itself a function of x.

Example

Suppose $f(x,y) = xy$ and $R = \{(x,y) \mid 1 \le x \le 2, x \le y \le x^2\}$. Compute

$$\int\int_R f(x,y)dx\,dy$$

$$
\begin{aligned}
\int\int_R f(x,y)dx\,dy &= \int\int_{\{(x,y)\ |\ 1\le x\le 2, x\le y\le x^2\}} xydx\,dy \\
&= \int_1^2 (\int_x^{x^2} xydy)dx \\
&= \int_1^2 (\frac{1}{2}xy^2|_{y=x}^{y=x^2})dx \\
&= \int_1^2 (\frac{1}{2}x(x^2)^2 - \frac{1}{2}x(x)^2)dx \\
&= \int_1^2 (\frac{1}{2}x^5 - \frac{1}{2}x^3)dx \\
&= (\frac{1}{12}x^6 - \frac{1}{8}x^4)|_{x=1}^{x=2} \\
&= (\frac{64}{12} - \frac{16}{8} - \frac{1}{12} + \frac{1}{8}) \\
&= \frac{27}{8}
\end{aligned}
$$

Another example

Compute

$$\int\int_{\{(x,y)\ |\ 0\le x\le 1, 0\le y\le e^x\}} e^x dx\,dy$$

Solution

$$\int\int_{\{(x,y)\ |\ 0\le x\le 1, 0\le y\le e^x\}} ye^x\,dx\,dy = \int_0^1 (\int_0^{e^x} e^x dy)dx$$
$$= \int_0^1 (ye^x|_{y=0}^{y=e^x})dx$$
$$= \int_0^1 e^{2x}dx$$
$$= (\frac{1}{2}e^{2x})|_{x=0}^{x=1}$$
$$= \frac{1}{2}(e^2-1)$$

Yet another example

Compute

$$\int\int_{\{(x,y)\ |\ 1\le x\le 4, \sqrt{x}\le y\le x^3\}} \frac{x}{y^2}\,dx\,dy$$

Solution

$$\begin{aligned}
\int\int_{\{(x,y)\ \mid\ 1\leq x\leq 4, \sqrt{x}\leq y\leq x^3\}} \frac{x^2}{y} dx\, dy &= \int_1^4 (\int_{\sqrt{x}}^{x^3} \frac{x}{y^2} dy) dx \\
&= \int_1^4 (\frac{-x}{y}|_{y=\sqrt{x}} y = x^3) dx \\
&= \int_1^4 (\sqrt{x} - \frac{1}{x^2}) dx \\
&= (\frac{2}{3} x^{\frac{3}{2}} + \frac{1}{x})|_{x=1} 4 \\
&= (\frac{16}{3} + \frac{1}{4} - \frac{2}{3} - 1) \\
&= \frac{47}{12}
\end{aligned}$$