12.5: Discrete infinite random variables

A discrete (infinite) random variable X is a random variable which may take a discrete though infinite set of possible values. For the sake of simplification, we assume that the possible values are the non-negative integers.

We present such a random variable by giving a sequence $p_0, p_1, p_2, \ldots$ of relative proportions. As with finite random variables, we presume that $0 \leq p_i \leq 1$ for all i and that $\sum_{i=0}^{\infty} p_i = 1$.

We regard p_i as the probability that X takes the value i.

Expected value and variance of discrete random variables

If X is a discrete random variable with $p_i = \Pr(X = i)$, then

$$E(X) = \sum_{n=0}^{\infty} n p_n$$

and

$$\mathrm{Var}(X) = (\sum_{n=0}^{\infty} n^2 p_n) - E(X)^2$$

Poisson random variable

The Poisson distribution with parameter λ is the discrete probability distribution with $p_n = \frac{\lambda^n}{n!} e^{-\lambda}$.

If X has this distribution, then

$$\begin{aligned} E(X) &= \sum_{n=0}^{\infty} n \frac{\lambda^n}{n!} e^{-\lambda} \\ &= \lambda e^{-\lambda} \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} \\ &= \lambda e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \\ &= \lambda e^{-\lambda} e^{\lambda} \\ &= \lambda \end{aligned}$$

A similar calculation shows that $\mathrm{Var}(X) = \lambda$.

Example

A certain website has an average of twenty hits per hour and the number of such hits is Poisson distributed. What is the probability that the website has ten or fewer hits in some hour?

Solution

$$\begin{aligned} \Pr(X \leq 10) &= \sum_{n=0}^{10} \frac{(20)^n}{n!} e^{-20} \\ &\approx .01081171882 \end{aligned}$$

Geometric random variable

Fix a number p with $0 \leq p \leq 1$. A discrete infinite random variable X is a geometric random variable with success probability p if the relative frequency of n is $p_n = (1-p)p^n$.

Using a geometric series one may compute that $E(X) = \frac{p}{1-p}$ while a Taylor series computation shows that $\mathrm{Var}(X) = \frac{p}{(1-p)^2}$.

Using geometric random variables

If one has a sequence of independent experiments where the probability of "success" being p, then the random variable which expresses the probability of n failures before the first success is geometrically distributed with success probability p.

Example

If one has a coin which comes up tails with a eighty percent probability, what is the probability that the first tails appears after a string of four heads?

Solution

We may model this situation with a geometric random variable for which "success" means obtaining heads. So, the success probability is $1 - .8 = 0.2$. The probability that the first tails is obtained after a string of four heads is then $p^4(1-p) = (0.2)^4(.8) = .00128$.