12.5: Discrete infinite random variables

A discrete (infinite) random variable X is a random variable which may take a discrete though infinite set of possible values. For the sake of simplification, we assume that the possible values are the non-negative integers.

We present such a random variable by giving a sequence p_0, p_1, p_2, \ldots of relative proportions. As with finite random variables, we presume that $0 \le p_i \le 1$ for all *i* and that $\sum_{i=0}^{\infty} p_i = 1$.

We regard p_i as the probability that X takes the value i.

Expected value and variance of discrete random variables

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If X is a discrete random variable with $p_i = \Pr(X = i)$, then

$$E(X) = \sum_{n=0}^{\infty} np_n$$

and

$$\operatorname{Var}(X) = \left(\sum_{n=0}^{\infty} n^2 p_n\right) - E(X)^2$$

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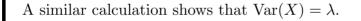
Poisson random variable

The Poisson distribution with parameter λ is the discrete probability distribution with $p_n = \frac{\lambda^n}{n!} e^{-\lambda}$. If X has this distribution, then

$$E(X) = \sum_{n=0}^{\infty} n \frac{\lambda^n}{n!} e^{-\lambda}$$
$$= \lambda e^{-\lambda} \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!}$$
$$= \lambda e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^m}{m!}$$
$$= \lambda e^{-\lambda} e^{\lambda}$$
$$= \lambda$$

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Example

A certain website has an average of twenty hits per hour and the number of such hits is Poisson distributed. What is the probability that the website has ten or fewer hits in some hour?

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Solution

$$\Pr(X \le 10) = \sum_{n=0}^{10} \frac{(20)^n}{n!} e^{-20}$$

\$\approx .01081171882\$

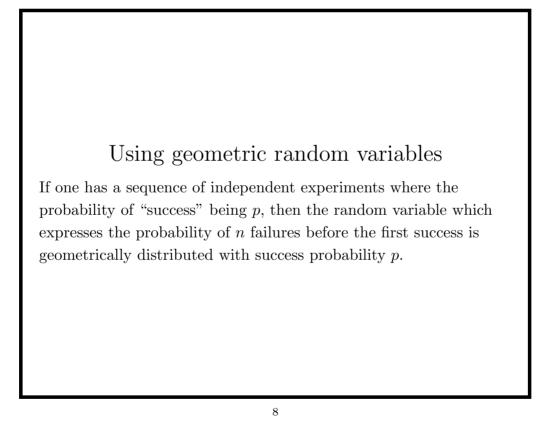
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Geometric random variable

Fix a number p with $0 \le p \le 1$. A discrete infinite random variable X is a geometric random variable with success probability p if the relative frequency of n is $p_n = (1-p)p^n$.

Using a geometric series one may compute that $E(X) = \frac{p}{1-p}$ while a Taylor series computation shows that $\operatorname{Var}(X) = \frac{p}{(1-p)^2}$.

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Example

If one has a coin which comes up tails with a eighty percent probability, what is the probability that the first tails appears after a string of four heads?

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Solution

We may model this situation with a geometric random variable for which "success" means obtaining heads. So, the success probability is 1 - .8 = 0.2. The probability that the first tails is obtained after a string of four heads is then $p^4(1-p) = (0.2)^4(.8) = .00128$.

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