12.3: Expected Value and Variance

If X is a random variable with corresponding probability density function f(x), then we *define* the expected value of X to be

$$E(X) := \int_{-\infty}^{\infty} x f(x) dx$$

We define the variance of X to be

$$\operatorname{Var}(X) := \int_{-\infty}^{\infty} [x - E(X)]^2 f(x) dx$$

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As with the variance of a discrete random variable, there is a simpler formula for the variance.

$$\begin{aligned} \operatorname{Var}(X) &= \int_{-\infty}^{\infty} [x - E(X)] f(x) dx \\ &= \int_{-\infty}^{\infty} [x^2 - 2x E(X) + E(X)^2] f(x) dx \\ &= \int_{-\infty}^{\infty} x^2 f(x) dx - 2E(X) \int_{-\infty}^{\infty} x f(x) dx \\ &+ E(X)^2 \int_{-\infty}^{\infty} f(x) dx \\ &= \int_{-\infty}^{\infty} x^2 f(x) dx - 2E(X) E(X) + E(X)^2 \times 1 \\ &= \int_{-\infty}^{\infty} x^2 f(x) dx - E(X)^2 \end{aligned}$$

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Interpretation of the expected value and the variance

The expected value should be regarded as the average value. When X is a discrete random variable, then the expected value of X is precisely the mean of the corresponding data.

The variance should be regarded as (something like) the average of the difference of the actual values from the average. A larger variance indicates a wider spread of values.

As with discrete random variables, sometimes one uses the *standard* deviation, $\sigma = \sqrt{\operatorname{Var}(X)}$, to measure the spread of the distribution instead.

Example

The uniform distribution on the interval [0, 1] has the probability density function

$$f(x) = \begin{cases} 0 \text{ if } x < 0 \text{ or } x > 1 \\ 1 \text{ if } 0 \le x \le 1 \end{cases}$$

Letting X be the associated random variable, compute E(X) and Var(X).

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Solution, completed

Hence,

$$Var(X) = \int_{-\infty}^{\infty} x^2 f(x) dx - E(X)^2$$
$$= \frac{1}{3} - \frac{1}{4}$$
$$= \frac{1}{12}$$

Another example

Let X be the random variable with probability density function

$$f(x) = \begin{cases} e^x \text{ if } x \le 0\\ 0 \text{ if } x > 0 \end{cases}$$

Compute E(X) and Var(X).

Solution

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Integrating by parts with u = x and $dv = e^x dx$, we see that $\int xe^x dx = xe^x - e^x + C$. Thus,

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx$$

=
$$\int_{-\infty}^{0} xe^{x}dx$$

=
$$\lim_{r \to -\infty} \int_{r}^{0} xe^{x}dx$$

=
$$\lim_{r \to -\infty} [-1 - re^{r} + e^{r}]$$

= 1

[We used L'Hôpital's rule to see that $\lim_{r \to -\infty} re^r = \lim_{r \to -\infty} \frac{r}{e^{-r}} = \lim_{r \to -\infty} \frac{1}{-e^{-r}} = 0.]$

We compute

$$\int x^2 e^x dx = x^2 e^x - 2 \int x e^x dx$$
$$= x^2 e^x - 2x e^x + 2e^x + C$$

Solution, continued

So,

$$\int_{-\infty}^{\infty} x^2 f(x) dx = \int_{-\infty}^{0} x^2 e^x dx$$
$$= \lim_{r \to -\infty} (2 - r^2 e^r + 2re^r - 2e^r)$$
$$= 2$$

This gives $Var(X) = 2 - 1^2 = 1$.

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One more example

Suppose that the random variable X has a cumulative distribution function

$$F(x) = \begin{cases} \sin(x) \text{ if } 0 \le x \le \frac{\pi}{2} \\ 0 \text{ if } x < 0 \text{ or } x > \frac{\pi}{2} \end{cases}$$

Compute E(X) and Var(X).

Solution

First, we must find the probability density function of X. Differentiating we find that the function

$$f(x) = \begin{cases} \cos(x) \text{ if } 0 \le x \le \frac{\pi}{2} \\ 0 \text{ otherwise} \end{cases}$$

is the derivative of F at all but two points. Thus, f(x) is a probability density function for X.

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Solution, finished

Integrating by parts, we compute

$$Var(X) = \int_0^{\frac{\pi}{2}} x^2 \cos(x) dx - E(X)^2$$

= $(x^2 \sin(x) - 2\sin(x) + 2x \cos(x))|_{x=0}^{x=\frac{\pi}{2}} - (\frac{\pi}{2} - 1)^2$
= $\frac{\pi^2}{4} - 2 - (\frac{\pi^2}{4} - \pi + 1)$
= $\pi - 3$