11.5: Taylor Series

A *power series* is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n$$

where each a_n is a number and x is a variable.

A power series defines a function $f(x) = \sum_{n=0}^{\infty} a_n x^n$ where we substitute numbers for x.

Note: The function f is only defined for those x with $\sum_{n=0}^{\infty} a_n x^n$ convergent.

Geometric series as a power series

For $|x| < 1$ we computed

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Taylor Series

If $f(x)$ is an infinitely differentiable function, then the *Taylor series of* $f(x)$ *at* a is the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Example

Compute the Taylor series of $f(x) = e^x$ at $a = 0$.

Solution

We know $f^{(n)}(x) = e^x$ for all $n \geq 0$. So $f^{(n)}(0) = 1$ and the Taylor series of $f(x) = e^x$ at $a = 0$ is

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

Example

Compute the Taylor series at $a = 1$ of $f(x) = \sqrt{x}$.

Solution

Write $f(x) = x^{\frac{1}{2}}$. Then $f'(x) = \frac{1}{2}x^{\frac{-1}{2}}$, $f''(x) = \frac{-1}{4}x^{\frac{-3}{2}}$, $f'''(x) = \frac{3}{8}x^{\frac{-5}{2}}$, $f^{(4)}(x) = \frac{-5}{16}x^{\frac{-7}{2}} = \frac{(-1)^{4+1}5\cdot 3\cdot 1}{2^4}x^{\frac{-7}{2}}$. In general, $f^{(n)}(x) = \frac{(-1)^{n+1}(2n-1)\cdot(2n-3)\cdots 3\cdot 1}{2^n}x^{\frac{-2n-1}{2}}$ so that $f^{(n)}(1) = \frac{(-1)^{n+1}(2n-1)\cdot(2n-3)\cdots 3\cdot 1}{2^n}$ and the Taylor series of $f(x) = \sqrt{x}$ at $a = 1$ is

$$1 + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(2n-1)(2n-3)\cdots 3\cdot 1}{n!2^n}(x-1)^n$$

Convergence of Taylor series

Given an infinitely differentiable function $f(x)$ with Taylor series (at a) $\sum_{n=0}^{\infty} b_n(x-a)^n$ either $\sum_{n=0}^{\infty} b_n(x-a)^n$ converges and is equal to $f(x)$ for every number x or there is a number R (called the *radius of convergence*) for which $\sum_{n=0}^{\infty} b_n(x-a)^n$ converges and is equal to $f(x)$ for $|x-a| < R$ while $\sum_{n=0}^{\infty} b_n(x-a)^n$ diverges for $|x-a| > R$.

Examples

- $$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$
  for all x.
- $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$
  for $|x| < 1$ and diverges for $|x| > 1$

Operations on Taylor series

Differentiation: If $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} a_n x^n$, then $f'(x) = \sum_{n=0}^{\infty} \frac{f^{(n+1)}(0)}{n!} x^n = \sum_{n=0}^{\infty} n a_n x^{n-1}$.

Integration: If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then $\int f(x) dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$.

Products: If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$, then $f(x) \cdot g(x) = \sum_{n=0}^{\infty} (\sum_{i=0}^{n} a_i b_{n-i}) x^n$.

Composition (monomial case): If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and m is a positive integer, then $f(x^m) = \sum_{n=0}^{\infty} a_n x^{nm}$.

Example

Compute the Taylor series at $a = 0$ of $\int e^{x^2} dx$.

Solution

We know $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$. So, $e^{x^2} = \sum_{n=0}^{\infty} \frac{1}{n!} x^{2n}$. Integrating, $\int e^{x^2} dx = \int \sum_{n=0}^{\infty} \frac{1}{n!} x^{2n} dx = C + \sum_{n=0}^{\infty} \frac{1}{n!(2n+1)} x^{2n+1}$.

That is, $\int e^{x^2} dx = C + x + \frac{1}{6}x^3 + \frac{1}{30}x^5 + \frac{1}{98}x^7 + \cdots$.

Example

Find the Taylor series at zero of $\frac{1+x^2}{1-x^3}$.

Solution

We know $\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n$ so that $\frac{1}{1-x^3} = \sum_{n=0}^{\infty} x^{3n}$.

Multiplying, $\frac{1+x^2}{1-x^3} = (1+x^2)\sum_{n=0}^{\infty} x^{3n} = \sum_{n=0}^{\infty}(x^{3n} + x^{3n+2}) =$
$1 + x^2 + x^3 + x^5 + x^6 + x^8 + x^9 + x^{11} + \cdots$