11.1: Taylor polynomials
The derivative as the first Taylor polynomial

If $f(x)$ is differentiable at a, then the function $p(x) = b + m(x - a)$ where $b = f(0)$ and $m = f'(x)$ is the “best” linear approximation to f near a.

For $x \approx a$ we have $f(x) \approx p(x)$.

Note that $f(a) = b = p(a)$ and $f'(a) = m = p'(a)$.

Higher degree Taylor polynomials

If $f(x)$ is a function which is n times differentiable at a, then the n^{th} Taylor polynomial of f at a is the polynomial $p(x)$ of degree (at most n) for which $f^{(i)}(a) = p^{(i)}(a)$ for all $i \leq n$.

Example

Compute the third Taylor polynomial of $f(x) = e^x$ at $a = 0$.

Solution

Write $p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$. We need to find c_0, c_1, c_2, and c_3 so that $p^{(i)}(0) = f^{(i)}(0)$ for $i = 0$, 1, 2, and 3.

In our case $f^{(i)}(x) = e^x$ for all $i \geq 0$ and $e^0 = 1$. So, $f^{(i)}(0) = 1$ for all i.

We compute $p'(x) = c_1 + 2c_2 x + 3c_3 x^2$, $p''(x) = 2c_2 + 6c_3 x$, and $p'''(x) = 6c_3$. Thus,

$1 = f^{(0)}(0) = p^{(0)}(0) = c_0$.

$1 = f^{(1)}(0) = p^{(1)}(0) = c_1$.

$1 = f^{(2)}(0) = p^{(2)}(0) = 2c_2$ so that $c_2 = \frac{1}{2}$.

Finally, $1 = f^{(3)}(0) = p^{(3)}(0) = 6c_3$ so that $c_3 = \frac{1}{6}$.

Thus, the third Taylor polynomial of $f(x) = e^x$ at $a = 0$ is $p(x) = \frac{1}{6}x^3 + \frac{1}{2}x^2 + x + 1$.

Another Example

Find the third Taylor polynomial of $f(x) = \ln(x)$ at $a = 1$.

A solution

As before, we write $p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$ and we find $p'(x) = c_1 + 2c_2 x + 3c_3 x^2$, $p''(x) = 2c_2 + 6c_3 x$, and $p'''(x) = 6c_3$.

Differentiating, $f'(x) = \frac{1}{x} = x^{-1}$, $f''(x) = -x^{-2}$, and $f'''(x) = 2x^{-3}$. Thus,

$0 = f^{(0)}(1) = p^{(0)}(1) = c_0 + c_1 + c_2 + c_3$

$1 = f^{(1)}(1) = p^{(1)}(1) = c_1 + 2c_2 + 3c_3$

$-1 = f^{(2)}(1) = p^{(2)}(1) = 2c_2 + 6c_3$

$2 = f^{(3)}(1) = p^{(3)}(1) = 6c_3$

Solving these equations, we find $c_3 = \frac{1}{3}$, $c_2 = \frac{-3}{2}$, $c_1 = 3$, and $c_0 = \frac{-11}{6}$.

That is, the third Taylor polynomial of $\ln(x)$ at $a = 1$ is $\frac{1}{3}x^3 - \frac{3}{2}x^2 + 3x - \frac{11}{6}$.

Another solution

We may write any polynomial of degree three as $p(x) = d_0 + d_1(x-1) + d_2(x-1)^2 + d_3(x-1)^3$.

Differentiating, we have $p'(x) = d_1 + 2d_2(x-1) + 3d_3(x-1)^2$, $p''(x) = 2d_2 + 6d_3(x-1)$, and $p'''(x) = 6d_3$.

So, $p(1) = d_0$, $p'(1) = d_1$, $p''(1) = 2d_2$, and $p'''(1) = 6d_3$.

Hence, if p is the third Taylor polynomial of $\ln(x)$ at $a = 1$, we have $d_0 = 0$, $d_1 = 1$, $d_2 = \frac{-1}{2}$, and $d_3 = \frac{1}{3}$.

That is, the third Taylor polynomial of $\ln(x)$ at $a = 0$ is $\frac{1}{3}(x-1)^3 - \frac{1}{2}(x-1)^2 + (x-1)$.

General formula for Taylor polynomials

If we write $p(x) = \sum_{i=0}^{n} d_i(x-a)^i$, then $p^{(j)}(x) = \sum_{i=j}^{n} \frac{i!}{(i-j)!} d_i(x-a)^{i-j}$ where $i! = i \cdot (i-1) \cdot (i-2) \cdots 2 \cdot 1$. (We define $0! = 1$ and $(i+1)! = (i+1) \cdot i!$.)

In particular, $p^{(j)}(a) = j! d_j$. So, if p is the n^{th} Taylor polynomial of f at a, we have $j! d_j = p^{(j)}(a) = f^{(j)}(a)$.

Thus, $d_j = \frac{1}{j!} f^{(j)}(a)$ or to put it another way, the n^{th} Taylor polynomial of f at a is $\sum_{j=0}^{n} \frac{1}{j!} f^{(j)}(a)(x-a)^j$.

Example

Compute the fifth Taylor polynomial of $f(x) = \sin(x)$ at $a = 0$.

Solution

We compute $f'(x) = \cos(x)$, $f''(x) = -\sin(x)$, $f'''(x) = -\cos(x)$, $f^{(4)}(x) = \sin(x)$, and $f^{(5)}(x) = \cos(x)$. Thus, $f^{(0)}(0) = 0$, $f^{(1)}(0) = 1$, $f^{(2)}(0) = 0$, $f^{(3)}(0) = -1$, $f^{(4)}(0) = 0$, and $f^{(5)}(0) = 1$.

We compute the first few factorials: $0! = 1$, $1! = 1 \cdot 0! = 1 \cdot 1 = 1$, $2! = 2 \cdot 1! = 2 \cdot 1 = 2$, $3! = 3 \cdot 2! = 3 \cdot 2 = 6$, $4! = 4 \cdot 3! = 4 \cdot 6 = 24$, and $5! = 5 \cdot 4! = 5 \cdot 24 = 120$.

Therefore, the fifth Taylor polynomial of $f(x) = \sin(x)$ at $a = 0$ is $\frac{1}{120}x^5 - \frac{1}{6}x^3 + x$.

Error estimates

If $f(x)$ is $(n+1)$ times differentiable between on the interval $[a,x]$ (or $[x,a]$ if $x < a$) and $p(x)$ is the n^{th} Taylor polynomial of f at a, then there is a number $a \le c \le x$ so that
$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$.

So, if we can find M so that $|f^{(n+1)}(y)| \le M$ whenever $a \le y \le x$, we would know that $|f(x) - p(x)| \le \frac{M}{(n+1)!}(x-a)^{n+1}$.

Example

Find a decimal approximation to e valid to the hundredths place.

Solution

We will find n so that if $p(x)$ is the n^{th} Taylor polynomial for $f(x) = e^x$ at $a = 0$, then $|e - p(1)| = |f(1) - p(1)| < \frac{1}{200}$.

We know that $f^{(n+1)}(x) = e^x$ and that on the interval from zero to one this function is bounded by 3.

Thus, $|e - p(1)| \leq \frac{3}{(n+1)!}(1-0)^{n+1} = \frac{3}{(n+1)!}$.

So, we want n so that $\frac{3}{(n+1)!} < \frac{1}{120}$ or what is the same thing $(n+1)! > 360$. If $n = 5$, then $(n+1)! = 6! = 720 > 360$.

Now, $p(x) = \frac{1}{120}x^5 + \frac{1}{24}x^4 + \frac{1}{6}x^3 + \frac{1}{2}x^2 + x + 1$. So, $e \approx \frac{1}{120} + \frac{1}{24} + \frac{1}{6} + \frac{1}{2} + 1 + 1 = \frac{1+5+20+60+120+120}{120} = \frac{326}{120} = \frac{163}{60} \approx$ 2.7166666666......

A better approximation to e

In point of fact, $e \approx$ 2.71828182845904523536028747135266249775724709369995957 49669676277240766303535475945713821785251664274274663919320030599218174135966290435729003342952605956307381323286279434907632338298807531952510190115738341879307021540891499348841675092447614606680822648001684774118537423454424371075390777449920695517027618386062613313845830007520449338265602976067371132007093287091274437470472306969772093101416928368190255151086574637721112523897844250569536967707854499699679468644549059879316368892300987931277361782154249992295763514822082698951936680331825288693984964651058209392398294887933203625094431.

Printed with line breaks and spaces in the digit string:

2.71828182845904523536028747135266249775724709369995957
49669676277240766303535475945713821785251664274274663919320030599218174135966290435729003342952605956307381323286279434907632338298807531952510190115738341879307021540891499348841675092447614606680822648001684774118537423454424371075390777449920695517027618386062613313845830007520449338265602976067371132007093287091274437470472306969772093101416928368190255151086574637721112523897844250569536967707854499699679468644549059879316368892300987931277361782154249992295763514822082698951936680331825288693984964651058209392398294887933203625094431.