Numerical Solutions of Differential Equations

A differential equation

$$y' = f(y, t)$$

may be approximated as a *difference equation*.

If $\Delta t \approx 0$, then

$$\begin{aligned} y(a + \Delta t) &\approx y(a) + y'(a)\Delta t \\ &= y(a) + f(y(a), a)\Delta t \end{aligned}$$

Euler's Method

Iterating the approximation $y(a + \Delta t) \approx y(a) + f(y(a), a)\Delta t$, we can numerically approximate solutions to initial value problems $y' = f(y, t)$ and $y(t_0) = y_0$.

That is, given that y satisfies the above initial value problem, to approximate $y(a)$, fix a positive integer n, set $\Delta = \frac{a - t_0}{n}$, and define $t_i := t_0 + i\Delta$ (for $0 \leq i \leq n$).

Euler's method, continued

We know that $y(t_0) = y_0$. Approximating, we have

$$\begin{aligned} y(t_1) &= y(t_0 + \Delta) \\ &\approx y(t_0) + \Delta y'(t_0) \\ &= y_0 + \Delta f(y_0, t_0) \\ &=: y_1 \end{aligned}$$

Repeating this process, we find that $y(t_2) \approx y_1 + \Delta f(y_1, t_1) =: y_2$, $\ldots$, $y(a) = y(t_n) \approx y_{n-1} + \Delta f(y_{n-1}, t_{n-1})$.

Example

Approximate the value of $y(1)$ when $y' = ty + 1$ and $y(0) = 0$ using $n = 2$.

Solution

Note that a symbolically solve $y' = ty + 1$ one must find an antiderivative to $e^{-\frac{1}{2}t^2}$.

Here $\Delta = \frac{1-0}{2} = 0.5$.

We compute

$$\begin{aligned} y(.5) &\approx 0 + (0.5)(0(0) + 1) \\ &= 0.5 \\ &= y_1 \end{aligned}$$

$$\begin{aligned} y(1) &\approx 0.5 + (0.5)((0.5)(0.5) + 1) \\ &= 0.5 + 0.5(1.25) \\ &= 0.5 + 0.625 \\ &= 1.125 \end{aligned}$$

Another Example

Approximate $y(1)$ when $y' = \sin(y)$ and $y(0) = .1$ using $n = 5$ subdivisions.

Obstructions to symbolic solutions

This time, our symbolic methods fail twice! To use the method of separation of variables, we would need to find an antiderivative of $\csc(y)$. Even if we were to succeed with this step, we would have to invert the function $\int \csc(y)dy$.

Solution

In this case, we compute mechanically.

$\Delta = \frac{1-0}{5} = 0.2$, $y_0 = 0.1$, and we wish to find $y_5 \approx y(1)$.

$$\begin{aligned} y(.2) &\approx y_0 + \Delta \sin(y_0) \\ &= 0.1 + (0.2)\sin(0.1) \\ &\approx 0.1000 + (0.2000)(0.0998) \\ &\approx 0.1200 \\ &=: y_1 \end{aligned}$$

$$\begin{aligned} y(.4) &\approx y_1 + \Delta \sin(y_1) \\ &= 0.1200 + (0.2000)\sin(0.1200) \\ &\approx 0.1429 \\ &=: y_2 \end{aligned}$$

$$\begin{aligned} y(.6) &\approx y_2 + \Delta \sin(y_2) \\ &= 0.1429 + (0.2)\sin(0.1429) \\ &\approx .1714 \\ &=: y_3 \end{aligned}$$

$$\begin{aligned} y(.8) &\approx y_3 + \Delta \sin(y_3) \\ &= .1714 + (0.2)\sin(.1714) \\ &\approx .2055 \\ &=: y_4 \end{aligned}$$

$$\begin{aligned} y(1) &\approx y_4 + \Delta \sin(y_4) \\ &= .2055 + (0.2)\sin(.2055) \\ &\approx .2463 \end{aligned}$$