# Section 10.3: Solving First-Order Linear Differential Equations: Integration Factors

A first order linear differential equation is a differential equation of the form

$$y' + a(t)y = b(t)$$



## Example

Solve the differential equation

$$y' + ty = 0$$

## Solution

In this case we can use the method of separation of variables.

If $y$ is constant, then $ty \equiv y' \equiv 0$ so that $y \equiv 0$.

Otherwise, we may express the equation as $\frac{y'}{y} = -t$. Let $C = y(0)$. Integrating with respect to $t$, we have

$$\begin{aligned} -\frac{1}{2}T^2 &= \int_0^T -t dt \\ &= \int_0^T \frac{y' dt}{y} \\ &= \int_C^{y(T)} \frac{dy}{y} \\ &= \ln|\frac{y(T)}{C}| \end{aligned}$$



## Solution, continued

(As our solution must be continous and cannot take the value zero, the signs of $y(T)$ and $C = y(0)$ must agree. So, we may drop the absolute value bars.)

Exponentiating both sides of this equation and multiplying by $C$, we obtain $y(T) = Ce^{\frac{-1}{2}T^2}$.

## Another Example

Solve the differential equation

$$y' + y = 10e^{-t}$$



## Solution

In this case, we cannot apply the separation of variables technique.

However, as $e^t$ is never equal to zero, the solutions to the original equation and to the equation

$$e^t y' + e^t y = 10$$

are the same.

Observe that

$$\frac{d}{dt}(e^t y) = e^t y' + e^t y$$

## Solution, continued

Thus, if our differential equation holds, we have $\frac{d}{dt}(e^t y) = 10$.

We integrate with respect to $t$.

$$
\begin{aligned}
e^T y(T) - y(0) &= e^t y(t)|_{t=0}^{t=T} \\
&= \int_0^T \frac{d}{dt}(e^t y) dt \\
&= \int_0^T 10 dt \\
&= 10T
\end{aligned}
$$

So, if we write $C = y(0)$, then we have $y(T) = 10e^{-T}T + Ce^{-T}$.



## A Third Example

Solve the differential equation

$$y' + \frac{1}{t} y = \cos(t)$$

## Solution

In this case, multiplying by $t$ we may express the equation as $ty' + y = t\cos(t)$. Using the product rule we check that $\frac{d}{dt}(ty) = ty' + y$.

We integrate this expression.

**Note:** The original equation is singular at $t = 0$ in the sense that the function $\frac{1}{t}$ is not defined. We need to take for the lower limit of integration some other constant. The number $\pi$ is a convenient choice in this case.



## Solution, continued

$$\begin{aligned} Ty(T) - \pi y(\pi) &= \int_\pi^T \frac{d}{dt}(ty)dt \\ &= \int_\pi^T t\cos(t)dt \\ &= t\sin(t) + \cos(t)|_{t=\pi}^{t=T} \text{ integrate by parts} \\ &\quad \text{with } u = t \text{ and } dv = \cos(t)dt \\ &= T\sin(T) + \cos(T) + 1 \end{aligned}$$

Write $C := y(\pi)$. Then we conclude that

$y(T) = \sin(T) + \frac{1}{T}(\cos(T) + 1 + \pi C)$.

## General Solution

In general, if $A'(t) = a(t)$, then

$$\begin{aligned} \frac{d}{dt}(e^{A(t)}y) &= e^{A(t)}y' + A'(t)e^{A(t)}y \\ &= e^{A(t)}(y' + a(t)y) \end{aligned}$$

Thus, a differential equation of the form $y' + a(t)y = b(t)$ may be expressed as $\frac{d}{dt}(e^{A(t)}y) = e^{A(t)}(y' + a(t)) = e^{A(t)}b(t)$.



## General solution, continued

So, if $\alpha$ is in the domain of the functions $a(t)$ and $b(t)$, we have

$$\begin{aligned} e^{A(T)}y(T) - e^{A(\alpha)}y(\alpha) &= \int_{\alpha}^{T} \frac{d}{dt}(e^{A(t)}y)dt \\ &= \int_{\alpha}^{T} e^{A(t)}b(t)dt \end{aligned}$$

Set $C := e^{A(\alpha)}y(\alpha)$, then $Y(T) = e^{-A(T)} \int_{\alpha}^{T} e^{A(t)}b(t)dt + Ce^{-A(T)}$.

# An example reconsidered

In solving the equation $y' + \frac{1}{t}y = \cos(t)$, we multiplied by $t$ and then observed that $\frac{d}{dt}(ty) = ty' + y = t(y' + \frac{1}{t}y)$.

In terms of the general solution, $a(t) = \frac{1}{t}$ and if $A(t) = \ln|t|$, then we have $A'(t) = a(t)$.

Note that $e^{A(t)} = e^{\ln|t|} = |t|$. So, multiplying by $t$ is the same as multiplying by $e^{A(t)}$ for $t > 0$.

Our general method gives

$$
\begin{aligned}
y(T) &= e^{-A(T)} \int_{\alpha}^{T} e^{A(t)} b(t)dt + Ce^{-A(T)} \\
&= \frac{1}{T} \int_{\alpha}^{T} t\cos(t)dt + \frac{\alpha y(\alpha)}{T}
\end{aligned}
$$

To finish, we must choose $\alpha$ and evaluate the above integral.