# Section 7.3: Maxima and Minima of Functions of Several Variables
## Review of Single Variable Case

If $f(x)$ is a (sufficiently differentiable) function of a single variable and $f$ has a relative minimum or maximum (generically an *extremum*) at $x = a$ then $f'(a) = 0$.

Recall that a function may have $f'(a) = 0$ without $a$ being an extremum.



## Extrema of a Function of a Single Variable: Examples

- $f(x) = x^2 + 1$ and $a = 0$
- $g(x) = e^x - x$ and $a = 0$
- $g(x) = x^3$ and $a = 0$

## First Derivative Test for Extrema of Functions of Two Variables

If $(a, b)$ is a relative extremum of $F(x, y)$, then $a$ is a relative extremum of $g(x) := F(x, b)$ and $b$ is a relative extremum of $h(y) := F(a, y)$. So,

$$0 = g'(a) = \frac{\partial F}{\partial x}|_{(a,b)}$$

and

$$0 = h'(b) = \frac{\partial F}{\partial y}|_{(a,b)}$$

(In fact, this test applies to functions in any number of variables.)



## Counterexamples

As with functions of a single variable, there may be points $(a, b)$ which are not relative extrema but for which $\frac{\partial F}{\partial x}|_{(a,b)} = 0 = \frac{\partial F}{\partial y}|_{(a,b)}$.

- $F(x, y) = x^3 y^2$
- $G(x, y) = \sin(x)\cos(y)$

## Second Derivative Test: One Variable

Recall that for a function of a single variable, one can look at the second derivative to test for concavity and thereby also the existence of a local minimum or maximum.

A (sufficiently smooth) function of one variable $f(x)$ has a relative extremum at $x = a$ if $f'(a) = 0$ and $f''(a) \neq 0$. If $f'(a) = 0$ and $f''(a) > 0$, then $a$ is a relative *minimum* and if $f'(a) = 0$ and $f''(a) < 0$, then $a$ is a relative *maximum*.



## Second Derivative Test: Two Variables

Given a function $F(x, y)$ of two variables we define a new function

$$D_F(x, y) := \frac{\partial F}{\partial x} \cdot \frac{\partial F}{\partial y} - (\frac{\partial^2 F}{\partial x \partial y})^2$$

If

- $\frac{\partial F}{\partial x}|_{(a,b)} = 0 = \frac{\partial F}{\partial y}|_{(a,b)}$,
- $D_F(a, b) > 0$, and
- $\frac{\partial^2 F}{\partial x^2}|_{(a,b)} \neq 0$;

then $F$ has a relative extremum at $(a, b)$ (maximum if $\frac{\partial^2 F}{\partial x^2}|_{(a,b)} < 0$ and minimum if this second derivative is positive).

## Second Derivative Test for Two Variables: No Extremum

Conversely, if

- $\frac{\partial F}{\partial x}|_{(a,b)} = 0 = \frac{\partial F}{\partial y}|_{(a,b)}$ and
- $D_F(a,b) < 0$;

then $F$ does *not* have a relative extremum at $(a,b)$.

When $D_F(a,b) = 0$, this test yields no information.



## Second Derivative Test: Example 1

Find the relative extrema of $f(x,y) = x^3 - y^2 - 3x + 2y$.

# Solution

$$\begin{aligned}
\frac{\partial f}{\partial x} &= 3x^2 - 3 \\
\frac{\partial f}{\partial y} &= 2 - 2y \\
\frac{\partial^2 f}{\partial x^2} &= 6x \\
\frac{\partial^2 f}{\partial x \partial y} &= 0 \\
\frac{\partial^2 f}{\partial y^2} &= -2 \\
D_f &= -12x
\end{aligned}$$



# Solution continued

The solutions to $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$ are $(-1, 1)$ and $(1, 1)$. We compute the $D_f(-1, 1) = 12 > 0$ and $D_f(1, 1) = -12 < 0$. Thus, the only potential relative extremum is at $(-1, 1)$.

We compute $\frac{\partial^2 f}{\partial x^2}|_{(-1,1)} = -6 < 0$. Thus, $(-1, 1)$ is a relative maximum.

## Second Derivative Test: Example 2

Find the extrema of $F(x, y) = ye^x - 3x - y + 2$.



## Solution

$$\frac{\partial F}{\partial x} = ye^x - 3$$
$$\frac{\partial F}{\partial y} = e^x - 1$$

Setting both of these equal to zero, we find $x = 0$ and $y = 3$.

## Solution, continued

$$
\begin{aligned}
\frac{\partial^2 F}{\partial x^2} &= ye^x \\
\frac{\partial^2 F}{\partial y^2} &= 0 \\
\frac{\partial^2 F}{\partial x \partial y} &= e^x \\
D_F &= -e^x
\end{aligned}
$$

As $D_F(0,3) = -1 < 0$, the point $(0,3,2)$ is not an extremum, there are no local extrema of $F$.