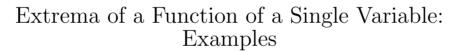
Section 7.3: Maxima and Minima of Functions of Several Variables Review of Single Variable Case

If f(x) is a (sufficiently differentiable) function of a single variable and f has a relative minimum or maximum (generically an *extremum*) at x = a then f'(a) = 0.

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Recall that a function may have f'(a) = 0 without a being an extremum.



- $f(x) = x^2 + 1$  and a = 0
- $g(x) = e^x x$  and a = 0
- $g(x) = x^3$  and a = 0

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## First Derivative Test for Extrema of Functions of Two Variables

If (a, b) is a relative extremum of F(x, y), then a is a relative extremum of g(x) := F(x, b) and b is a relative extremum of h(y) := F(a, y). So,

$$0 = g'(a) = \frac{\partial F}{\partial x}|_{(a,b)}$$

and

$$0 = h'(b) = \frac{\partial F}{\partial y}|_{(a,b)}$$

(In fact, this test applies to functions in any number of variables.)

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### Counterexamples

As with functions of a single variable, there may be points (a, b) which are not relative extrema but for which  $\frac{\partial F}{\partial x}|_{(a,b)} = 0 = \frac{\partial F}{\partial y}|_{(a,b)}.$ 

- $F(x,y) = x^3y^2$
- $G(x,y) = \sin(x)\cos(y)$

#### Second Derivative Test: One Variable

Recall that for a function of a single variable, one can look at the second derivative to test for concavity and thereby also the existence of a local minimum or maximum.

A (sufficiently smooth) function of one variable f(x) has a relative extremum at x = a if f'(a) = 0 and  $f''(a) \neq 0$ . If f'(a) = 0 and f''(a) > 0, then a is a relative minimum and if f'(a) = 0 and f''(a) < 0, then a is a relative maximum.

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#### Second Derivative Test: Two Variables

Given a function F(x, y) of two variables we define a new function

$$D_F(x,y) := \frac{\partial F}{\partial x} \cdot \frac{\partial F}{\partial y} - (\frac{\partial^2 F}{\partial x \partial y})^2$$

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• 
$$\frac{\partial F}{\partial x}|_{(a,b)} = 0 = \frac{\partial F}{\partial y}|_{(a,b)},$$

- $D_F(a,b) > 0$ , and
- $\frac{\partial^2 F}{\partial x^2}|_{(a,b)} \neq 0;$

then F has a relative extremum at (a, b) (maximum if  $\frac{\partial^2 F}{\partial x^2}|_{(a,b)} < 0$ and minimum if this second derivative is positive).

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# Second Derivative Test for Two Variables: No Extremum

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Conversely, if

- $\frac{\partial F}{\partial x}|_{(a,b)} = 0 = \frac{\partial F}{\partial y}|_{(a,b)}$  and
- $D_F(a,b) < 0;$

then F does not have a relative extremum at (a, b). When  $D_F(a, b) = 0$ , this test yields no information.

Second Derivative Test: Example 1

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Find the relative extrema of  $f(x, y) = x^3 - y^2 - 3x + 2y$ .

Solution			
$rac{\partial f}{\partial x}$	=	$3x^2 - 3$	
$rac{\partial f}{\partial y}$	=	2-2y	
$rac{\partial^2 f}{\partial x^2}$	=	6x	
$rac{\partial^2 f}{\partial x \partial y}$	=	0	
$rac{\partial^2 f}{\partial y^2}$	=	-2 -12x	
$D_f$	=	-12x	
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## Solution continued

The solutions to  $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$  are (-1, 1) and (1, 1). We compute the  $D_f(-1, 1) = 12 > 0$  and  $D_f(1, 1) = -12 < 0$ . Thus, the only potential relative extremum is at (-1, 1).

We compute  $\frac{\partial^2 f}{\partial x^2}|_{(-1,1)} = -6 < 0$ . Thus, (-1,1) is a relative maximum.

# Second Derivative Test: Example 2

Find the extrema of  $F(x, y) = ye^x - 3x - y + 2$ .

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Solution

$$\frac{\partial F}{\partial x} = ye^x - 3$$
$$\frac{\partial F}{\partial y} = e^x - 1$$

Setting both of these equal to zero, we find x = 0 and y = 3.

Solution, continued

$$\frac{\partial^2 F}{\partial x^2} = y e^x$$
$$\frac{\partial^2 F}{\partial y^2} = 0$$
$$\frac{\partial^2 F}{\partial x \partial y} = e^x$$
$$D_F = -e^x$$

As  $D_F(0,3) = -1 < 0$ , the point (0,3,2) is not an extremum, there are no local extrema of F.