Section 10.2: Separation of variables

The method of *separation of variables* applies to differential equations of the form

$$y' = p(t)q(y)$$

where $p(t)$ and $q(x)$ are functions of a single variable.

Example

Find the general solution to the differential equation

$$y' = ty^2$$

Solution

Any constant solution to this equation would have $0 \equiv ty^2$ so that $y \equiv 0$.

Avoiding the constant solution, we may divide both sides of the equation by y^2 and then we solve:

$$\begin{aligned}
\frac{T^2}{2} &= \int_0^T t dt \\
&= \int_0^T \frac{y'}{y^2} dt \\
&= \int_{y(0)}^{y(T)} y^{-2} dy \\
&= \frac{-1}{y}\Big|_{y(0)}^{y(T)} \\
&= \frac{1}{y(0)} - \frac{1}{y(T)}
\end{aligned}$$

So, if we set $C := y(0)$, we have $y = \frac{2C}{2-Ct^2}$.

General procedure

To solve the differential equation $y' = p(t)q(y)$:

- Find the constant solutions by solving for $q(c) = 0$.
- Find $P(t)$ an antiderivative of $p(t)$ and $Q(x)$ an antiderivative of $\frac{1}{q(x)}$.
- Write $c = y(0)$. We find $Q(y) - Q(c) = \int_0^y \frac{dy}{q(y)} = \int_0^T p(t)dt = P(T) - P(0)$.
- Solve for y.

Example

Find the general solution of $y' = y\sin(t) - \sin(t)$

Solution

We begin by rewriting the equation at $y' = (y-1)\sin(t)$.

The only constant solution is $y \equiv 1$.

Integrating, we find that $\ln(|y-1|)$ is an antiderivative of $\frac{1}{y-1}$ while $-\cos(t)$ is an antiderivative of $\sin(t)$.

Let $C = y(0)$. Then we have $\ln(|y-1|) - \ln(|C-1|) = 1 - \cos(t)$.

Adding $\ln(|C-1|)$ to both sides and applying the exponential function, we conclude that $|y-1| = |C-1|e^{1-\cos(t)}$.

As the solution y must be continuous, the signs of $y-1$ and $C-1$ agree. Thus, $y = 1 + (C-1)e^{1-\cos(t)}$.

Note: In this case the constant solution has the same form.

Yet another example

Find the general solution to the differential equation $y' = ty + 1$

No elementary solution!

The method of separation of variables does not apply as the function $ty + 1$ *cannot* be written as the product of a function of y by a function of t.

Scholium: Using Taylor series expansions (a topic which we shall discuss next month), one can compute an expression for solutions to the equation $y' = ty + 1$.

Another Example

Find the general solution to the equation

$$y' = \frac{\sec^2 t}{y + 1}$$

Solution

There are no constant solutions as $\frac{1}{x+1}$ is never zero. Note, however, that we cannot have $y(t) = -1$ as the differential equation would require y to be nondifferentiable at such a point.

As before, we set $C = y(0)$. Multiplying by $y + 1$ and integrating, we find

$$\begin{aligned} \tan(T) &= \int_0^T \sec^(t)dt \\ &= \int_0^T (y(t)+1)y'(t)dt \\ &= \int_C^{y(T)} (y+1)dy \\ &= \frac{1}{2}y(T)^2 + y(T) - \frac{1}{2}C^2 - C \end{aligned}$$

Solution, continued

So, y satisfies the equation

$$y^2 + 2y - C^2 - 2C - 2\tan(t) = 0$$

From the quadratic formula, we compute that

$$\begin{aligned} y &= \frac{-2 \pm \sqrt{2^2 - 4(-C^2 - 2C - 2\tan(t))}}{2} \\ &= -1 \pm \sqrt{1 + C^2 + 2C + 2\tan(t)} \\ &= -1 \pm \sqrt{(C+1)^2 + 2\tan(t)} \end{aligned}$$

Undefined solutions, multiple solutions

- For each choice of $C \neq -1$, we found **two** solutions to the initial value problem $y' = \frac{\sec^2(t)}{y+1}$, namely $y = -1 + \sqrt{(C+1)^2 + 2\tan(t)}$ and $y = -1 - \sqrt{(C+1)^2 + 2\tan(t)}$. However, only $y = -1 + \sqrt{(C+1)^2 + 2\tan(t)}$ satisfies $y(0) = C$.
- Strictly speaking, there is no solution with $y(0) = -1$, but there is a solution having $\lim_{t \to O^+} y(t) = -1$.
- **No** solution to the differential equation is defined for all values of t. If $t \ll 0$ so that $\tan(t) < \frac{-(C+1)^2}{2}$, then $\sqrt{(C+1)^2 + 2\tan(t)}$ is not a real number.

Another Example

Find a function y satisfying $y(0) = 5$ and $y' = \frac{t}{e^y}$.

Solution

As the exponential function never attains the value zero, there are no constant solutions to this differential equation. Multiplying both sides of the equation by e^y and integrating, we obtain:

$$\begin{aligned}
\frac{1}{2}T^2 &= \int_0^T tdt \\
&= \int_0^T e^{y(t)}y'(t)dy \\
&= \int_5^{y(T)} e^y dy \\
&= e^{y(T)} - e^5
\end{aligned}$$

Solution, continued

Addding e^5 to both sides of this equation and taking the natural logarithm, we compute

$$\begin{aligned}
y &= \ln(e^y) \\
&= \ln(e^5 + \frac{1}{2}t^2)
\end{aligned}$$