Section 10.2: Separation of variables

The method of *separation of variables* applies to differential equations of the form

y' = p(t)q(y)

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where p(t) and q(x) are functions of a single variable.

Example

Find the general solution to the differential equation

$$y' = ty^2$$

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Any constant solution to this equation would have $0 \equiv ty^2$ so that $y \equiv 0$.

Avoiding the constant solution, we may divide both sides of the equation by y^2 and then we solve:

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$$\begin{aligned} \frac{T^2}{2} &= \int_0^T t dt \\ &= \int_0^T \frac{y'}{y^2} dt \\ &= \int_{y(0)}^{y(T)} y^{-2} dy \\ &= \frac{-1}{y} |_{y(0)}^{y(T)} \\ &= \frac{1}{y(0)} - \frac{1}{y(T)} \end{aligned}$$

So, if we set $C := y(0)$, we have $y = \frac{2C}{2-Ct^2}$.

General procedure

To solve the differential equation y' = p(t)q(y):

- Find the constant solutions by solving for q(c) = 0.
- Find P(t) an antiderivative of p(t) and Q(x) an antiderivative of $\frac{1}{q(x)}$.

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- Write c = y(0). We find $Q(y) - Q(c) = \int_0^y \frac{dy}{q(y)} = \int_0^T p(t)dt = P(T) - P(0).$
- Solve for y.

Example

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Find the general solution of $y' = y \sin(t) - \sin(t)$

We begin by rewriting the equation at $y' = (y - 1)\sin(t)$. The only constant solution is $y \equiv 1$. Integrating, we find that $\ln(|y - 1|)$ is an antiderivative of $\frac{1}{y-1}$ while $-\cos(t)$ is an antiderivative of $\sin(t)$. Let C = y(0). Then we have $\ln(|y - 1|) - \ln(|C - 1|) = 1 - \cos(t)$. Adding $\ln(|C - 1|)$ to both sides and applying the exponential function, we conclude that $|y - 1| = |C - 1|e^{1-\cos(t)}$. As the solution y must be continuous, the signs of y - 1 and C - 1agree. Thus, $y = 1 + (C - 1)e^{1-\cos(t)}$. Note: In this case the constant solution has the same form.

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Yet another example

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Find the general solution to the differential equation y' = ty + 1

No elementary solution!

The method of separation of variables does not apply as the function ty + 1 cannot be written as the product of a function of y by a function of t.

Scholium: Using Taylor series expansions (a topic which we shall discuss next month), one can compute an expression for solutions to the equation y' = ty + 1.

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Another Example

Find the general solution to the equation

$$y' = \frac{\sec^2 t}{y+1}$$

There are no constant solutions as $\frac{1}{x+1}$ is never zero. Note, however, that we cannot have y(t) = -1 as the differential equation would require y to be nondifferentiable at such a point.

As before, we set C = y(0). Multiplying by y + 1 and integrating, we find

$$\tan(T) = \int_{0}^{T} \sec^{(t)} dt$$

= $\int_{0}^{T} (y(t) + 1)y'(t)dt$
= $\int_{C}^{y(T)} (y+1)dy$
= $\frac{1}{2}y(T)^{2} + y(T) - \frac{1}{2}C^{2} - C$

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Solution, continued

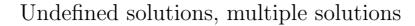
So, y satisfies the equation

$$y^2 + 2y - C^2 - 2C - 2\tan(t) = 0$$

From the quadratic formula, we compute that

$$y = \frac{-2 \pm \sqrt{2^2 - 4(-C^2 - 2C - 2\tan(t))}}{2}$$

= $-1 \pm \sqrt{1 + C^2 + 2C + 2\tan(t)}$
= $-1 \pm \sqrt{(C+1)^2 + 2\tan(t)}$



- For each choice of $C \neq -1$, we found **two** solutions to the initial value problem $y' = \frac{\sec^2(t)}{y+1}$, namely $y = -1 + \sqrt{(C+1)^2 + 2\tan(t)}$ and
 - $y = -1 \sqrt{(C+1)^2 + 2\tan(t)}$. However, only
 - $y = -1 + \sqrt{(C+1)^2 + 2\tan(t)}$ satisfies y(0) = C.
- Strictly speaking, there is no solution with y(0) = -1, but there is a solution having $\lim_{t\to O^+} y(t) = -1$.
- No solution to the differential equation is defined for all values of t. If $t \ll 0$ so that $\tan(t) < \frac{-(C+1)^2}{2}$, then $\sqrt{(C+1)^2 + 2\tan(t)}$ is not a real number.

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Another Example

Find a function y satisfying y(0) = 5 and $y' = \frac{t}{e^y}$.

As the exponential function never attains the value zero, there are no constant solutions to this differential equation. Multiplying both sides of the equation by e^y and integrating, we obtain:

$$\frac{1}{2}T^2 = \int_0^T t dt$$
$$= \int_0^T e^{y(t)}y'(t)dy$$
$$= \int_5^{y(T)} e^y dy$$
$$= e^{y(T)} - e^5$$

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Solution, continued

Addding e^5 to both sides of this equation and taking the natural logarithm, we compute

$$y = \ln(e^y)$$
$$= \ln(e^5 + \frac{1}{2}t^2)$$