Section 10.1: Solutions of Differential Equations

An (ordinary) differential equation is an equation involving a function and its derivatives.

That is, for functions $P(x_0, x_1, \ldots, x_n)$ and $Q(x_0, \ldots, x_n)$ of $n+1$ variables, we say that the function $f(t)$ (of one variable) satisfies the differential equation

$$P(y, y', \ldots, y^{(n)}) = Q(f(t), \ldots, f^{(n)}(t))$$

if

$$P(f(t), f'(t), \ldots, f^{(n)}(t)) \equiv 0$$

Examples

- The function $f(t) = e^t$ satisfies the differential equation $y' = y$.
- The constant function $g(t) \equiv 5$ satisfies the differential equation $y' = 0$.
- The functions $h(t) = \sin(t)$ and $k(t) = \cos(t)$ satisfy the differential equation $y'' + y = 0$.
- The function $\ell(t) = \ln(t)$ satisfies $-(y')^2 = y''$.

Initial value problems

An *initial value problem* is a differential equation given together with some requirements on the value of the function (or possibly some of its derivatives) at certain points.

Example

Find a function $f(t)$ which satisfies $f'(t) \equiv f(t)$ and $f(0) = 30$.

That is, solve the initial value problem $y' = y$ and $y(0) = 30$.

A solution

We know that if $f(t) = Ce^t$, for some constant C, then $f'(t) = Ce^t = f(t)$. So, such a function is a solution to the differential equation $y' = y$. To solve the initial value problem we need to specify C.

Evaluating,

$$\begin{aligned} 30 &= f(0) \text{ as } f \text{ satisfies the initial value condition} \\ &= Ce^0 \\ &= C \end{aligned}$$

So, $f(t) = 30e^t$ solves the initial value problem.

Another example

Solve the initial value problem $y' = \cos(t)$ and $y(0) = 1$.

Solution

If $f'(t) = \cos(t)$, then, by definition, $f(t)$ is an antiderivative of $\cos(t)$. We know $\int \cos(t)dt = \sin(t) + C$. Thus, $f(t) = \sin(t) + C$ for *some* constant C. Evaluating at zero, we have $1 = f(0) = \sin(0) + C = C$. Thus, the solution to this initial value problem is $f(t) = \sin(t) + 1$.

Constant solutions

In general, a solution to a differential equation is a function. However, the function could be a constant function.

For example, all solutions to the equation $y' = 0$ are constant. There are nontrivial differential equations which have some constant solutions.

Example

Find constant solutions to the differential equation

$$y'' - (y')^2 + y^2 - y = 0$$

Solution

$y = c$ is a constant, then $y' = 0$ (and, *a fortiori* $y'' = 0$). So, we would need $c^2 - c = 0$. That is, $y \equiv 1$ or $y \equiv 0$. Substituting, one checks that these values satify the differential equation.

Differential equations presented as narrative problems
Newton's Law of Cooling

Newton's Law of Cooling states that the rate of cooling of an object is proportional to the difference between its temperature and the ambient temperature.

How are we to express this law in terms of differential equations?

Solution

- Newton's Law expresses a fact about the temperature of an object over time. Express the temperature of the object at time t as $y(t)$.
- "rate of cooling" refers to a rate of change of the temperature, *ie* to $y'(t)$. The word "cooling" suggests that the derivative is negative.
- To say that some quantity is *proportional* to another is to say that there is some number α so that the first quantity is equal to α times the second, independent of the time.

Solution, continued

- In this formulation of the law, the ambient temperature is taken to be fixed; call it A. The difference between the object's temperature and the ambient temperature is then $y(t) - A$.
- Putting these together, we obtain the differential equation $y' = \alpha(y - A)$.

Using Newton's Law of Cooling

Suppose that an object initially having a temperature of 20° is placed in a large temperature controlled room of 80° and one hour later the object has a temperature of 35°. What will its temperature be after three hours?

Solution

We must solve the initial value problem $y' = \alpha(y-80)$, $y(0) = 20$, and $y(1) = 35$ where α is some unknown constant.

Clearly, the solution $y \equiv 80$ to the differential equation $y' = \alpha(y-80)$ does not satisfy the initial value conditions. So, we may divide by $y - 80$, obtaining

$$\frac{y'}{y-80} = \alpha$$

Solution, continued

Integrating with respect to t from 0 to time T and substituting $u = y - 80$ (so that $du = y'dt$) we have

$$\begin{aligned} \alpha T &= \int_0^T \alpha dt \\ &= \int_0^T \frac{y'dt}{y-80} \\ &= \int_{-60}^{y(T)-80} \frac{du}{u} \\ &= \ln|u| \Big|_{-60}^{y(T)-80} \\ &= \ln(60) - \ln(80 - y(T)) \end{aligned}$$

So $\ln(80 - y(T)) = \ln(60) - \alpha T$.

Solution, continued

Exponentiating both sides, we obtain

$$80 - y(T) = 60e^{-\alpha T}$$

Evaluating at $T = 1$, we have

$$45 = 80 - 35 = 60e^{-\alpha}$$

So, $\alpha = \ln(\frac{3}{4})$.

Thus, $y(T) = 80 - 60e^{T\ln(\frac{3}{4})} = 80 - 60(\frac{3}{4})^T$. So that $y(3) = 80 - 60(\frac{3}{4})^3 = 80 - 60(\frac{27}{64}) = 54\frac{11}{16}$.

Logistic equations

A *logistic equation* is a differential equation of the form $y' = \alpha y(y - M)$ for some constants α and M.

The logistic equation has the constant solutions $y \equiv 0$ and $y \equiv M$ and the nonconstant solution

$$y(t) = \frac{M}{1 + (\frac{M - y(0)}{y(0)})e^{\alpha M t}}$$