Integration Techniques
Example

Integrate

$$\int x^3 \ln(x) dx$$

A solution

Let $u = x^4$ so that $du = 4x^3 dx$. Note that $4\ln(x) = \ln(x^4)$. So,

$$\begin{aligned}
\int x^3 \ln(x) dx &= \frac{1}{16} \int \ln(x^4)(4x^3) dx \\
&= \frac{1}{16} \int \ln(u) du \\
&= \frac{1}{16}(u \ln(u) - u) + C \\
&= \frac{1}{4} x^4 \ln(x) - \frac{1}{16} x^4 + C
\end{aligned}$$

Another solution

Set $u = \ln(x)$ and $dv = x^3 dx$. So that $du = \frac{dx}{x}$ and $v = \frac{1}{4}x^4$.

Then,

$$\begin{aligned}
\int x^3 \ln(x) dx &= \int u dv \\
&= uv - \int v du \\
&= \frac{1}{4}x^4 \ln(x) - \int \frac{1}{4}x^4 \frac{1}{x} dx \\
&= \frac{1}{4}x^4 \ln(x) - \frac{1}{4}\int x^3 dx \\
&= \frac{1}{4}x^4 \ln(x) - \frac{1}{16}x^4 + C
\end{aligned}$$

Another example

Compute

$$\int \frac{4x}{\sqrt{x-1}} dx$$

Solution

Set $u = 4x$ and $dv = \frac{dx}{\sqrt{x-1}} = (x-1)^{-\frac{1}{2}}dx$ so that $du = 4dx$ and $v = 2\sqrt{x-1}$. Then

$$\begin{aligned}\int \frac{4x}{\sqrt{x-1}}dx &= \int udv \\ &= 8x\sqrt{x-1} - 8\int \sqrt{x-1}dx \\ &= 8x\sqrt{x-1} - \frac{16}{3}(x-1)^{\frac{3}{2}} + C\end{aligned}$$

Another example

Compute

$$\int \frac{\ln(\ln(x))}{x\ln(x)}dx$$

Solution

Set $u = \ln(\ln(x))$ so that $du = \frac{dx}{x\ln(x)}$.

Then

$$\begin{aligned}\int \frac{\ln(\ln(x))}{x\ln(x)}dx &= \int du \\ &= u + C \\ &= \ln(\ln(x)) + C\end{aligned}$$

Another example

Compute

$$\int_{-\pi}^{\pi} x\sin(x^4)dx$$

Solution

If $f(x) = x\sin(x^4)$, then $f(-x) = -f(x)$. Thus,
$\int_{-\pi}^{0} x\sin(x^4)dx = -\int_{0}^{\pi} x\sin(x^4)dx$. So, $\int_{-\pi}^{\pi} x\sin(x^4)dx = 0$.