Section 9.3: Evaluation of Definite Integrals
The Fundamental Theorem of Calculus, recalled

If $F'(x) = f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

An example

Evaluate

$$\int_1^{10} \frac{(\ln(x))^4}{x} dx$$

A solution

Substitute $u = \ln(x)$ so that $du = \frac{dx}{x}$.

Thus,

$$\begin{aligned} \int \frac{(\ln(x))^4}{x} dx &= \int u^4 du \\ &= \frac{1}{5} u^5 + C \\ &= \frac{1}{5} (\ln(x))^5 + C \end{aligned}$$

Solution, continued

Thus,

$$\begin{aligned} \int_1^{10} \frac{(\ln(x))^4}{x} dx &= \frac{1}{5} (\ln(10))^5 - \frac{1}{5} (\ln(1))^5 \\ &= \frac{1}{5} (\ln(10))^5 \end{aligned}$$

Substituting the limits as well

If we substitute $u = g(x)$ so that $\int f(x)dx = \int h(u)du$, then

$$\int_a^b f(x)dx = \int_{g(a)}^{g(b)} h(u)du$$

Indeed, if $H'(u) = h(u)$ and $f(x) = h(g(x))g'(x)$, then $(H \circ g)'(x) = H'(g(x))g'(x) = h(g(x))g'(x) = f(x)$. Thus, by the fundamental theorem of calculus,

$$\begin{aligned} \int_{g(a)}^{g(b)} h(u)du &= H(g(b)) - H(g(a)) \\ &= \int_a^b f(x)dx \end{aligned}$$

Example

Evaluate

$$\int_1^2 x^2 e^{x^3} dx$$

Solution

Set $u = x^3$, then $du = 3x^2 dx$. So,

$$\begin{aligned}\int_1^2 x^2 e^{x^3} dx &= \frac{1}{3}\int_{u=1^3}^{u=2^3} e^u du \\ &= \frac{1}{3} e^u |_{u=1}^{u=8} \\ &= \frac{1}{3}e^8 - \frac{1}{3}e\end{aligned}$$

Another example

Evaluate

$$\int_0^{\frac{\pi}{4}} \tan(x) dx$$

Solution

Write $\tan(x) = \frac{\sin(x)}{\cos(x)}$ Set $u = \cos(x)$. Then $du = -\sin(x)dx$.

Thus,

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \tan(x)dx &= -\int_{u=1}^{u=\frac{\sqrt{2}}{2}} \frac{du}{u} \\
&= -\ln(u)\Big|_{u=1}^{u=\frac{\sqrt{2}}{2}} \\
&= -\ln(1) + \ln(\frac{\sqrt{2}}{2}) \\
&= \ln(2^{\frac{1}{2}}) - \ln(2) \\
&= \frac{1}{2}\ln(2) - \ln(2) \\
&= -\frac{1}{2}\ln(2)
\end{aligned}$$

Example

Compute

$$\int_0^{\pi} x\sin(x^2) + x^2\cos(x)dx$$

Solution

Break the integral into two summands, $\int_0^\pi x\sin(x^2)dx$ and $\int_0^\pi x^2\cos(x)dx$. We compute these separately. For the former, set $u = x^2$ so that $du = 2xdx$. Thus,

$$\begin{aligned}\int_{x=0}^{x=\pi} x\sin(x^2)dx &= \frac{1}{2}\int_{u=0}^{u=\pi^2} \sin(u)du \\ &= \frac{-1}{2}\cos(u)|_{u=0}^{u=\pi^2} \\ &= \frac{1}{2}(1-\cos(\pi^2))\end{aligned}$$

Solution, continued

For the latter part, set $u = x^2$ and $dv = \cos(x)dx$. Then, $du = 2xdx$ and $v = \sin(x)$. So that

$$\int x^2\cos(x)dx = x^2\sin(x) - 2\int x\sin(x)dx$$

Set $w = x$ and $dy = \sin(x)dx$. Then, $dw = dx$ and $y = -\cos(x)$. So,

$$\int x\sin(x)dx = -x\cos(x) + \int \cos(x)dx = -x\cos(x) + \sin(x) + C$$

Combining these,

$$\int x^2\cos(x)dx = x^2\sin(x) + 2x\cos(x) - 2\sin(x) + C$$

Solution, continued

Evaluating,

$$\begin{aligned}
\int_0^\pi x^2\cos(x)dx &= x^2\sin(x) + 2x\cos(x) - 2\sin(x)|_{x=0}^{x=\pi} \\
&= \pi^2\sin(\pi) + 2\pi\cos(\pi) \\
&\quad -2\sin(\pi) - 0^2\sin(0) - 2(0)\cos(0) + 2\sin(0) \\
&= 2\pi
\end{aligned}$$

So,

$$\int_0^\pi (x\sin(x^2) + x^2\cos(x))dx = \frac{1}{2}(1-\cos(\pi^2)) + 2\pi$$