Section 9.2: Integrations by Parts
A Test Problem

Perform the following indefinite integration.

$$\int x \sin(x) dx$$

Answer

$$\int x \sin(x) dx = \sin(x) - x\cos(x) + C$$

Product Rule Recalled

$$(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$$

Inverting the Chain Rule: Integration by Substitution

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

Formalism of Integration by Parts

Often, one finds two functions u and v so that the integrand may be written as udv where $dv = v'(x)dx$. If we succeed in so doing, then from the equality

$$u(x)v'(x) = u(x)v(x) - v(x)u'(x)$$

we see that if $f(x) = u(x)v'(x)$, then

$$\int f(x)dx = \int u(x)v'(x)dx = \int udv = uv - \int vdu$$

If $\int vdu$ is easier to evaluate than is $\int f(x)dx$, then the method succeeds.

An integral revisited

Take $u = x$ and $v = -\cos(x)$ so that $dv = \sin(x)dx$ and $du = dx$. Then,

$$\begin{aligned}\int x\sin(x)dx &= \int udv \\ &= uv - \int vdu \\ &= -x\cos(x) + \int \cos(x)dx \\ &= -x\cos(x) + \sin(x) + C\end{aligned}$$

Example

Integrate:

$$\int xe^x dx$$

Solution

Take $u = x$ and $v = e^x$ so that $du = dx$ and $dv = e^x dx$.

$$\begin{aligned} \int xe^x dx &= \int udv \\ &= uv - \int vdu \\ &= xe^x - \int e^x dx \\ &= xe^x - e^x + C \end{aligned}$$

Large Another example

Integrate

$$\int e^x \cos(x) dx$$

Solution

Set $u = e^x$ and $v = \sin(x)$ so that $du = e^x$ and $dv = \cos(x)dx$.

Then

$$\begin{aligned}
\int e^x \cos(x) dx &= \int u dv \\
&= uv - \int v du \\
&= e^x \sin(x) - \int e^x \sin(x) dx \\
&= e^x \sin(x) + \int (-e^x \sin(x)) dx
\end{aligned}$$

Solution, continued

Set $w = e^x$ and $y = \cos(x)$ so that $dw = e^x dx$ and $dy = -\sin(x)dx$.

Then

$$\begin{aligned} \int -e^x \sin(x)dx &= \int w dy \\ &= wy - \int y dw \\ &= e^x \cos(x) - \int e^x \cos(x)dx \end{aligned}$$

Substituting, we have

$$\int e^x \cos(x)dx = e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)dx$$

Adding $\int e^x \cos(x)dx$ and dividing by 2 gives

$$\int e^x \cos(x)dx = \frac{e^x}{2}(\sin(x) + \cos(x)) + C$$

A third example

Integrate

$$\int \ln(x)dx$$

Solution

Set $u = \ln(x)$ and $v = x$ so that $du = \frac{dx}{x}$ and $dv = dx$.

$$\begin{aligned}
\int \ln(x)dx &= \int udv \\
&= uv - \int vdu \\
&= x\ln(x) - \int x\frac{1}{x}dx \\
&= x\ln(x) - \int dx \\
&= x\ln(x) - x + C
\end{aligned}$$

A final example

Integrate

$$\int \frac{xe^{2x}}{x^2 + 4x + 4} dx$$

Solution

Note that $\frac{1}{4x^2+4x+1} = (2x+1)^{-2}$. Via the substitution $w = 2x + 1$ (with $dw = 2dx$) we find that

$$\begin{aligned}
\int \frac{dx}{4x^2 + 4x + 1} &= \int (2x+2)^{-2} dx \\
&= \frac{1}{2} \int w^{-2} dw \\
&= \frac{-1}{2} w^{-1} + C \\
&= \frac{-1}{4x+2} + C
\end{aligned}$$

Solution, continued

Set $u = xe^{2x}$ and $v = \frac{-1}{4x+2}$ (so that $dv = \frac{dx}{x^2+4x+4}$ and $du = e^{2x} + 2xe^{2x} = (2x+1)e^{2x}$).

$$\begin{aligned}
\int \frac{xe^{2x}}{4x^2+4x+1}dx & \\
&= \int udv \\
&= uv - \int vdu \\
&= \frac{-xe^{2x}}{4x+2} + \int \frac{(2x+1)e^{2x}}{4x+2}dx \\
&= \frac{-xe^{2x}}{4x+2} + \int \frac{1}{2}e^{2x}dx \\
&= \frac{-xe^{2x}}{4x+2} + \frac{1}{4}e^{2x} + C
\end{aligned}$$