Chapter 9: Integration Techniques
Section 9.1: Integrations by Substitution

Perform the following indefinite integration.

$$\int x\sqrt{x^2+1}dx$$

Answer

$$\int x\sqrt{x^2+1}dx = \frac{1}{3}(x^2+1)^{\frac{3}{2}} + C$$

Chain Rule Recalled

$$\frac{d}{dx}(f(g(x)) = \frac{df}{du}|_{u=g(x)}\frac{dg}{dx}$$

Inverting the Chain Rule: Integration by Substitution

$$\int f'(g(x))g'(x)dx = f(g(x)) + C$$

Formalism of Integration by Substitution

Often, one writes the substitution as $u = g(x)$ and $du = g'(x)dx$ and attempts to write the integrand as $f(u)du = f(g(x))g'(x)dx$. One is then charged with integrating $\int f(u)du = F(u) + C$. We conclude that
$\int f(g(x))g'(x)dx = \int f(u)du = F(u) + C = F(g(x)) + C$.

An integral revisited

Making the substitution $u = x^2 + 1$, so that $du = 2xdx$ we have

$$\int x\sqrt{x^2+1}dx = \frac{1}{2}\int \sqrt{u}du$$

Writing $\sqrt{u} = u^{\frac{1}{2}}$, we find

$$\int \sqrt{u}du = \frac{2}{3}u^{\frac{3}{2}} + C$$

Thus,

$$\int x\sqrt{x^2+1}dx = \frac{1}{3}(x^2+1)^{\frac{3}{2}} + \widetilde{C}$$

Example

Integrate:

$$\int xe^{x^2}dx$$

Solution

Set $u = x^2$. Then $du = 2xdx$.

$$\begin{aligned}\int xe^{x^2}dx &= \frac{1}{2}\int e^{x^2}(2x)dx \\ &= \frac{1}{2}\int e^u du \\ &= \frac{1}{2}e^u + C \\ &= \frac{1}{2}e^{x^2} + C\end{aligned}$$

Another Example

Integrate

$$\int \sin(x)\cos(x)dx$$

Solution

Set $u = \sin(x)$ so that $du = \cos(x)dx$.

$$\begin{aligned} \int \sin(x)\cos(x)dx &= \int u du \\ &= \frac{1}{2}u^2 + C \\ &= \frac{1}{2}\sin^2(x) + C \end{aligned}$$

A third example

Integrate

$$\int \frac{\sqrt{\ln(x)}}{x} dx$$

Solution

Set $u = \ln(x)$ so that $du = \frac{1}{x} dx$.

$$\begin{aligned}
\int \frac{\sqrt{\ln(x)}}{x} dx &= \int \sqrt{u} du \\
&= \int u^{\frac{1}{2}} du \\
&= \frac{2}{3} u^{\frac{3}{2}} + C \\
&= \frac{2}{3} (\ln(x))^{\frac{3}{2}} + C
\end{aligned}$$

A final example

Integrate

$$\int \frac{\sin(x)}{\cos^3(x)} dx$$

Solution

Set $u = \tan(x)$. Then $du = \sec^2(x)dx$.

$$\begin{aligned} \int \frac{\sin(x)}{\cos^3(x)} dx &= \int \tan(x) \sec^2(x) dx \\ &= \int u du \\ &= \frac{1}{2} u^2 + C \\ &= \frac{1}{2} \tan^2(x) + C \end{aligned}$$

An alternate solution

Set $v = \cos(x)$ so that $dv = -\sin(x)dx$.

$$\begin{aligned} \int \frac{\sin(x)}{\cos^3(x)}dx &= -\int v^{-3}du \\ &= \frac{1}{2}v^{-2} + C' \\ &= \frac{1}{2}\sec^2(x) + C' \end{aligned}$$

Error?

Why are these answers different?

Answer

$$\tan^2(x) + 1 = \sec^2(x)$$

So, $C = \frac{1}{2} + C'$.