Convergence of Series

Disclaimer: This handout contains a summary of the various techniques for determining whether or not a series $\sum a_n$ converges. It may *not* be used during any quizzes or exams, but is intended for study purposes only. You will be responsible for all of the methods and techniques discussed in lectures and in the text, even if they do not appear on this handout.

1. Divergence Test, *11.2*. *Always check this test first, just in case... it's quick and easy.*

If $\lim_{n\to\infty} a_n \neq 0$, or if $\lim_{n\to\infty} a_n$ does not exist, then $\sum_{n=1}^{\infty} a_n$ diverges.

Examples: $\sum_{n=1}^{\infty} \frac{n^2+1}{n^2}$ $\sum_{n=1}^{\infty} \arctan(n)$ $\sum_{n=1}^{\infty} n \sin\left(\frac{1}{n}\right)$

2. Geometric Series, *11.2*.

$\sum_{n=1}^{\infty} ar^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \geq 1$ (when it converges, it is $\frac{a}{1-r}$).

Examples: $\sum_{n=1}^{\infty} \frac{3^{n-1}}{2^n}$ $\sum_{n=1}^{\infty} \frac{1}{e^n}$ $\sum_{n=1}^{\infty} \cos^n(1)$

3. Integral Test *11.3*.

You can only use this test when f is a continuous, positive, decreasing function. Then:
if $\int_1^{\infty} f(x)dx$ converges, then $\sum_{n=1}^{\infty} f(n)$ converges; if $\int_1^{\infty} f(x)dx$ diverges, then $\sum_{n=1}^{\infty} f(n)$ diverges.

Examples: $\sum_{n=1}^{\infty} \frac{1}{n\left(\ln(n)\right)^2}$ $\sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$ $\sum_{n=1}^{\infty} \frac{n^2+1}{\sqrt{n^3+3n-1}}$

4. p-Series, *11.3*. *This follows from the integral test, but it saves you time if you use it instead.*

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Examples: $\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$ $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

5. Alternating Series Test *11.5*.

If $a_n \geq 0 \; \forall \; n$, then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges if $\lim_{n\to\infty} a_n = 0$ and diverges if $\lim_{n\to\infty} a_n \neq 0$.

Examples: $\sum_{n=1}^{\infty} \frac{(-3)^n}{n2^n}$ $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n}$ $\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right)$

6. Comparison Test *11.4. Use this test when you have a series that looks "almost" like a p-series or geometric series.*

To use this test, you must have $b_n \geq a_n > 0$. Then:
if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges; or if $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ also diverges.

Examples: $\sum_{n=1}^{\infty} \frac{n!}{(n+1)!}$ $\sum_{n=1}^{\infty} \frac{1}{3+2^n}$ $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^5+5}}$

6.5. Limit Comparison Test *11.4. You can use this instead of the comparison test... it is often much easier to use and you are less likely to make a mistake.*

To use this test, you must have $a_n, b_n > 0$. Then:
If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then the series $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} a_n$ both converge or both diverge.

Examples: $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ $\sum_{n=1}^{\infty} \frac{n-1}{ne^n}$ $\sum_{n=1}^{\infty} \frac{n+\frac{1}{n}}{\sqrt{n^4+n^3-1}}$

7. Ratio Test *11.6. This is the most useful test... use it early and often, esp. with n!.*

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum_{n=1}^{\infty} a_n$ converges. If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.
If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, you must use a different test.. this one is inconclusive.

Examples: $\sum_{n=1}^{\infty} \frac{n}{e^n}$ $\sum_{n=1}^{\infty} \frac{10^{n-1}}{n!}$ $\sum_{n=1}^{\infty} \frac{n^2-1}{n!}$

8. Root Test *11.6. The root test is useful only when you have and exponent of n in the terms.*

If $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$, then $\sum_{n=1}^{\infty} a_n$ converges. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.
If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, you must use a different test.. this one is inconclusive.

Examples: $\sum_{n=2}^{\infty} \frac{1}{\ln(n)^n}$ $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ $\sum_{n=1}^{\infty} r^n$