# Practice Problems: Negating Statements

October 12, 2006

Each of the following statements is actually a definition you will encounter if you continue in your mathematical studies. Write down the negation of each statement.

1. (What it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be *bounded*). There exists a constant $M > 0$ such that $|f(x)| \leq M$ for all $x \in \mathbb{R}$.

2. (What it means for a sequence $x_n$ of real numbers to converge to 3). For every $\varepsilon > 0$, there exists a positive integer $N$ such that $|x_n - 3| < \varepsilon$ for all positive integers $n$ such that $n \geq N$.

3. (What it means for a sequence $x_n$ of real numbers to be convergent) There exists a real number $c$ such that for every $\varepsilon > 0$, there exists a positive integer $N$ such that for every positive integer $n \geq N$,
$$|x_n - c| < \varepsilon.$$

4. (What it means for a sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ to *converge pointwise* to another function $f : \mathbb{R} \to \mathbb{R}$). There exists a function $f : \mathbb{R} \to \mathbb{R}$ such that for all $x_0 \in \mathbb{R}$ and all $\varepsilon > 0$ there exists a positive integer $N$ such that for all $n \geq N$,
$$|f_n(x_0) - f(x_0)| < \epsilon.$$

5. (What it means for a sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ to *converge uniformly* to another function $f : \mathbb{R} \to \mathbb{R}$) There exists a function $f : \mathbb{R} \to \mathbb{R}$ such that for all $\varepsilon > 0$ there exists a positive integer $N$ such that for all $n \geq N$ and all $x \in \mathbb{R}$,
$$|f_n(x) - f(x)| < \epsilon.$$

6. (What it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be *continuous at the point* $x_0 \in \mathbb{R}$.) For every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ whenever $x \in \mathbb{R}$ such that $|x - x_0| < \delta$.

7. (What it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be *continuous*) For every $x_0 \in \mathbb{R}$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ whenever $x \in \mathbb{R}$ such that $|x - x_0| < \delta$.

8. (What it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be *uniformly continuous*). For every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in \mathbb{R}$ such that $|x - y| < \delta$.

9. (What it means for a function $T : V \to V$ on a vector space $V$ to be *linear*.) For all vectors $v, w \in V$ and all constants $a, b \in \mathbb{R}$,

$$T(av + bw) = aT(v) + bT(w).$$

10. (What it means for a collection of operators $\{f_\alpha \mid \alpha \in \Lambda\}$ on a Banach Space $X$ (whatever that is) to be a *uniformly bounded*). There exists a constant $M > 0$ such that for every $\alpha \in \Lambda$ and all $x \in X$,

$$\|T_\alpha(x)\| \leq M\|x\|.$$

11. (What it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be a *contraction*) There exists a constant $\gamma < 1$ such that

$$|f(x) - f(y)| \leq \gamma |x - y|$$

for all $x, y \in \mathbb{R}$.

12. (What it means for a subset $\{v_1, v_2, \dots, v_k\}$ of a vector space $V$ to be *linearly independent*.) For all constants $a_1, a_2, \dots, a_k \in \mathbb{R}$, if

$$\sum_{j=1}^{k} a_j v_j = 0$$

then

$$a_j = 0 \quad \text{for all} \quad j \in \{1, 2, \dots, k\}.$$

13. (What it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be *differentiable* at a point $x_0 \in \mathbb{R}$.)

    There exists a real number $c$ such that for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - x| < \delta$ then

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - c \right| < \varepsilon.$$

14. (What it means for a subset $U$ of $\mathbb{R}$ to be *open*.) For all $x \in U$ there exists $\varepsilon > 0$ such that $y \in U$ for all $y \in \mathbb{R}$ such that $|x - y| < \varepsilon$.

15. (What it means for a point $y \in \mathbb{R}$ to be a *cluster point* of a subset $U \subseteq \mathbb{R}$.)

    For all $\varepsilon > 0$ there exists $x \in U$ such that $|x - y| < \varepsilon$.

16. (What it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be *Lipschitz*.) There exists a constant $L$ such that for all $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| \leq L|x - y|.$$

17. (What it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be *periodic*.) There exists a real number $p > 0$ such that $f(x + p) = f(x)$ for all $x \in \mathbb{R}$.

18. (What it means for a subset $D$ of $\mathbb{R}$ to be *dense*.) For every $x \in \mathbb{R}$ and all $\varepsilon > 0$, there exists $y \in D$ such that $|x - y| < \varepsilon$.

19. (What it means for a subset $W$ of $\mathbb{R}$ to be *nowhere dense*.) For every $x \in \mathbb{R}$ and all $\varepsilon > 0$, there exists $y \in \mathbb{R}$ such that $|x - y| < \varepsilon$ and a positive constant $\delta > 0$ such that $|y - z| < \delta$ implies that $y \notin W$ for all $z \in \mathbb{R}$.