Math 74 - Practice Final Exam

December 14, 2006, 12:30pm-3:30pm in 31 Evans

Name: ____

This is a closed book, closed notes exam. Calculators are allowed, so long as they are "simple" calculators– no graphing calculators are allowed. If you are unsure if your calculator qualifies as "simple", ask your instructor. You have three hours to complete the exam. To receive full credit, write legibly, show your work, and write proofs in complete sentences. Unless otherwise stated, you must prove all of your claims. If you need more space, use the back of the page of the problem on which you are working.

Problem	Points	Your Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
7	20	
8	20	
9	20	
Total	160	

1. (a) Write down the negation of the statement "There exists a constant $\gamma > 0$ such that for all $n \in \mathbb{N}$, if m is an integer greater than n, then either $f(m) = 0$ or $f(m) > f(n) + \gamma$.

(b) Prove or provide a counterexample: For all sets A, B, and C, the set

$$A \cap (B \cup C) = (A \cap B) \cup C.$$

2. Prove each of the following or provide a counterexample.

(a) The equation $\gcd(m, n) = \gcd(3m - n, 2n - 5m)$ holds for all $m, n \in \mathbb{N}^*$.

(b) The equation $\gcd(m, n) < \gcd(3m + n, 9m + 3n)$ holds for all $m, n \in \mathbb{N}^*$.

(c) The equation $\gcd(m, n) = \gcd(m + n, m - 3n)$ holds for all $m, n \in \mathbb{N}^*$.

(d) The equation $\gcd(m, n) \neq \gcd(2m, 2n + 2m)$ holds for all $m, n \in \mathbb{N}^*$.

3. (a) Developing the appropriate context, define the terms *injective*, *surjective*, and *bijective*.

(b) Suppose that X, Y are sets and $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is the identity function on X. That is, $g \circ f = \mathrm{id}_X$. Must f be injective? Must f be surjective? Bijective? Give a proof or a counterexample for each.

(c) Must g be injective? Surjective? Bijective? Give a proof or a counterexample for each.

4. (a) Define what a *prime* is.

(b) State the Principle of Strong Induction.

(c) Show that every integer greater than 1 can be written as a product of one or more primes.

5. (a) Developing the appropriate context, define the terms *image* and *preimage*.

(b) Prove the following proposition.

Proposition. Suppose that $f : X \to Y$ is a map, $A \subset X$ and $B \subset Y$. Then B and $f(A)$ are disjoint if and only if A and$f^{-1}[B]$ are disjoint.

proof.

6. (a) State the Chinese Remainder Theorem.

(b) Does the system

$$\begin{cases} 3x \equiv 20 \pmod{38} \\ x \equiv 56 \pmod{1000} \end{cases}$$

have any integer solutions? If so, calculate the set of all solutions.

7. (a) Calculate $\gcd(138, 222)$ and find integers a and b such that $\gcd(138, 222) = 138a + 222b$.

(b) Find the smallest positive integer k for which there are integers x and y satisfying

$$138x + 222y = 10^{161616} + k.$$

Prove that you have found the smallest such k (you do not have to find a pair x and y which satisfy the equation above).

8. Find a real number A such that for every integer $n \geq 1$,

$$\sum_{k=1}^{n} (2k-1)^2 = A \cdot n(2n-1)(2n+1).$$

Prove your claim.

9. (a) State the Euclidean Algorithm.

(b) Suppose that a and m are relatively prime positive integers. If $n \in \mathbb{N}$ such that a divides mn, show that a divides n.

(c) Suppose that a and m are relatively prime integers, and that n is another integer relatively prime to a. Show that a is relatively prime to the product mn.