Name: Solutions

Quiz 4

Math 74

September 27, 2006

1. Developing the appropriate context, write down the definition of **injective**.

Let $f: X \to Y$ be a function. Then we say that f is <u>injective</u> if $f(x_1) = f(x_2)$ implies that $x_1 = x_2$ for all $x_1, x_2 \in X$.

2. Suppose that $f: X \to Y$ and $g: Y \to Z$ are injective. Prove that the function $h: X \to Z$ given by $h(x) = g(f(x))$ is also injective.

Choose $x_1, x_2 \in X$ such that $h(x_1) = h(x_2)$. Then $g(f(x_1)) = g(f(x_2))$. Since g is injective, this implies that $f(x_1) = f(x_2)$. Since f is injective, this implies that $x_1 = x_2$.

3. Show by induction that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

for every positive integer n.

Define $A = \{n \in \mathbb{N} \mid \sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)\}$. Clearly $1 \in A$ since $\sum_{k=1}^{1} k^2 = 1 = \frac{1(1+1)(2\cdot 1+1)}{6}$. Suppose that $m \in A$. Then $\sum_{k=1}^{m} k^2 = \frac{1}{6}m(m+1)(2m+1)$.

Thus $\sum_{k=1}^{m+1} k^2 = (m+1)^2 + \sum_{k=1}^{m} k^2 = (m+1)^2 + \frac{1}{6}m(m+1)(2m+1)$

$= \frac{(m+1)}{6} \cdot \left[6(m+1) + m(2m+1)\right] = \frac{1}{6}(m+1)\cdot(2m^2+7m+6) = \frac{1}{6}(m+1)(2m+3)(m+2)$

$= \frac{1}{6}(m+1)(m+2)(2(m+1)+1)$, so $m+1 \in A$. By induction, $A = \mathbb{N}$.