Math 74 - (SAMPLE) Midterm #1

Friday, October 6, 2006, 3:00pm-4:00pm

Name: Solutions

This is a closed book, closed notes exam. Calculators are not allowed. You have fifty minutes to complete the exam. To receive full credit, write legibly and in complete sentences. If you need more space, use the back of the page of the problem on which you are working.

Problem	Points	Your Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	80	

1. (a) Write down the negation of the following statement: "For every $\epsilon > 0$ there exists a positive integer N such that $|x_m - x_n| \le \epsilon$ for all positive integers m and n such that $m \ge N$ and $n \ge N$."

There exists $\varepsilon > 0$ such that for every $N \in \mathbb{N}^*$ there exist positive integers m and n such that $m \geq N$ and $n \geq N$ but $|x_m - x_n| > \varepsilon$.

(b) Give a proof or a counterexample to the following statement: Let A, B, and C be sets. Then $(A \backslash B) \backslash C = (A \backslash C) \backslash (B \backslash C)$.

<u>proof.</u> Let $x \in (A \backslash B) \backslash C$. Then $x \in A \backslash B$ and $x \notin C$. Thus $x \in A$, $x \notin B$, and $x \notin C$. Thus $x \in A \backslash C$ and $x \notin B \backslash C$. Thus $x \in (A \backslash C) \backslash (B \backslash C)$. We have demonstrated that $(A \backslash B) \backslash C \subseteq (A \backslash C) \backslash (B \backslash C)$.

Now select $y \in (A \backslash C) \backslash (B \backslash C)$. Then $y \in A \backslash C$ and $y \notin B \backslash C$. Then $y \in A$ and $y \notin C$. Since $y \notin B \backslash C$ and $y \notin C$, it follows that $y \notin B$. Thus $y \in A \backslash B$, and so $y \in (A \backslash B) \backslash C$. This shows that $(A \backslash C) \backslash (B \backslash C) \subseteq (A \backslash B) \backslash C$. ∎

2. (a) Prove the following statement or provide a counterexample: If $f : X \to Y$ is a map and $B_1, B_2 \subseteq Y$, then $B_1 \subseteq B_2$ if and only if $f^{-1}[B_1] \subseteq f^{-1}[B_2]$.

The statement is false, as the following counterexample demonstrates. Let $f : \{1\} \to \{1,2\}$ be given by $1 \mapsto 1$. Let $B_1 = \{2\}$ and $B_1 = \{1\}$. Then clearly $B_1 \nsubseteq B_2$, but $f^{-1}[B_1] = \emptyset$ and $f^{-1}[B_2] = \{1\}$, so $f^{-1}[B_1] \subseteq f^{-1}[B_2]$.

(b) Prove the following statement or provide a counterexample: If $f : X \to Y$ is a map, $A \subseteq X$ and $B \subseteq Y$, then $f(A) = B$ if and only if $f^{-1}[B] = A$.

This statement is false. Consider the map

$$f : \{1,2\} \to \{1\}.$$

Let $A = \{1\}$ and $B = \{1\}$. Then $f(A) = B$, but $f^{-1}[B] = \{1,2\} \neq A$.

3. (a) Developing the appropriate context, define the terms *injection*, *surjection* and *bijection*.

See the lecture notes.

(b) Is the function $g : \mathbb{Z} \to \mathbb{Z}$ defined by $g(n) = 5 - 2n$ injective? Surjective? Bijective?

We will show that g is injective. Suppose $m, n \in \mathbb{Z}$ such that $g(m) = g(n)$. Then $5-2m = 5-2n$, so $2m = 2n$, so $m = n$. Thus g is injective.

We will show that $g(n) \neq 0$ for all $n \in \mathbb{Z}$. For if $g(n) = 0$, then we would have $2n = 5$, or $n = 5/2 \notin \mathbb{Z}$, a contradiction. Thus g is not surjective, and thus not bijective, either.

4. (a) State the Well-Ordering Principle.

See the lecture notes

(b) Suppose that $n \in \mathbb{N}^*$. Show that there exists a unique positive integer k such that $k(k-1) < n \le k(k+1)$.

Let $G = \{m \in \mathbb{N}^* \mid n \le m(m+1)\}$. Then since $n+1 > 1$, $n \le n(n+1)$, and so $n \in G$. Thus G is non empty. By the Well-Ordering Principle, G has a least element. Let $k = \min G$. Then, since $k \in G$, we have that $n \le k(k+1)$. Since $k-1 \notin G$, we have that

$k(k-1) < n$. Hence $k(k-1) < n \le k(k+1)$.

Suppose that $\tilde{k} \in \mathbb{N}^*$ also satisfies $\tilde{k}(\tilde{k}-1) < n \le \tilde{k}(\tilde{k}+1)$. Then $\tilde{k} \in G$, so $k \le \tilde{k}$. If $\tilde{k} > k$, then $\tilde{k} \ge k+1$, and so $n \le k(k+1) \le \tilde{k}(\tilde{k}-1)$, a contradiction. Hence $\tilde{k} \le k$, which implies that $k = \tilde{k}$.

5. ((**Bonus Problem**) Suppose that $f : X \to Y$ and $g : Y \to Z$ such that the function $h : X \to Z$ given by $h(x) = g(f(x))$ is bijective.

(a) Must f be injective? Surjective?

– We will show that f is injective. Suppose $f(x_1) = f(x_2)$ for some $x_1, x_2 \in X$. Then $h(x_1) = g(f(x_1)) = g(f(x_2)) = h(x_2)$. Since h is injective, $x_1 = x_2$.

– As the following example shows, f need not be a surjection. Let $X = \{1\} = Z$, $Y = \{1, 2\}$, and $f : X \to Y$ be given by $1 \mapsto 1$, $g : Y \to Z$ be given by $y \mapsto 1$. Then obviously h is the function $\{1\} \to \{1\}$ given by $1 \mapsto 1$, a bijection. But f is not surjective, and g is not injective.

(b) Must g be injective? Surjective?

– The example above shows that g need not be ~~surje~~ injective.

– We will show that g is surjective. Pick $z \in Z$. Since h is surjective, we can find $x \in X$ such that $z = h(x)$. But then $z = g(f(x))$.