Solutions to Homework Assignment 4
Math 74, Fall 2006

October 6, 2006

1.

Definition 1. A natural number n is a **perfect square** if there exists an integer k such that $n = k^2$.

Theorem 2. *If $n \in \mathbb{N}$ then $\sqrt{n} \in \mathbb{Q}$ if and only if n is a perfect square.*

Proof. Clearly if $n \in \mathbb{N}$ is a perfect square, then $\sqrt{n} \in \mathbb{N} \subset \mathbb{Q}$. Suppose, on the other hand, that $n \in \mathbb{N}$ is not a perfect square. Define

$$B = \left\{k \in \mathbb{N} \mid k\sqrt{n} \in \mathbb{Z}\right\}.$$

We will show that B is nonempty. Proceeding by way of contradiction, suppose that B is not empty. By the Well-Ordering Principle, B has a least element. Let $b = \min B$. Then $b\sqrt{n} \in \mathbb{Z}$ since $b \in B$. Now choose an integer m such that

$$m < \sqrt{n} \leq m + 1.$$

By the Well-Ordering Principle, it is obvious that such an integer m exists, and that it is unique. Also, since n is not a perfect square, $\sqrt{n} \neq m + 1$, and so $\sqrt{n} < m + 1$. Now define

$$k = b(\sqrt{n} - m).$$

Notice that k is an integer, since $k = b\sqrt{n} - bm$ is the difference of two integers. Also, our choice of b guarantees that $0 < \sqrt{n} - m < 1$. So we conclude that

$$0 < k < b.$$

Since $k\sqrt{n} = bn - m(b\sqrt{n})$ is the difference of two integers, and is therefore an integer, we deduce that $k \in B$. But $k < b$ contradicts the fact that $k = \min B$. □

2. (a)

Proposition 3. *The set*

$$\{5m + 7n \mid m, n \in \mathbb{N}\} = \{0, 5, 7, 10, 12, 14, 15, 17, 19, 20, 21, 22\} \cup \{k \in \mathbb{N} \mid k \geq 24\}.$$

Proof. Denote the set $\{5m+7n \mid m,n \in \mathbb{N}\}$ by $\mathcal{F}$. It is easy enough to check that

$$\mathcal{F} \cap \{n \in \mathbb{N} \mid 1 \leq n \leq 28\} = \{0,5,7,10,12,14,15,17,19,20,21,22,24,25,26,27,28\}$$

Suppose that $\{24,\ldots,k\} \subseteq \mathcal{F}$ for some $k \geq 28$. Then $k-4 \geq 28-4 = 24$, so $k-4 \in \mathcal{F}$. Thus there are integers m and n such that

$$k-4 = 5m+7n.$$

Thus

$$k+1 = k-4+5 = 5m+7n+5 = 5(m+1)+7n,$$

and so $k+1 \in \mathcal{F}$. The result now follows by strong induction. $\square$

(b)

Proposition 4. *The set*

$$\{n \in \mathbb{N} \mid n \geq 32\} \subseteq \{5m+9n \mid m,n \in \mathbb{N}\}.$$

Proof. Denote the set $\{5m+9n \mid m,n \in \mathbb{N}\}$ by $\mathcal{G}$. It is easy to check that $32,33,34,35,36 \in \mathcal{G}$. Suppose that $\{32,\ldots,k\} \subseteq \mathcal{G}$ for some $k \geq 36$. Then $k-4 \geq 32$, so $k-4 \in \mathcal{G}$. Thus there exists $m,n \in \mathbb{N}$ such that

$$k-4 = 5m+9n.$$

Thus

$$k+1 = 5(m+1)+9n,$$

so $k+1 \in \mathcal{G}$. The result now follows by induction. $\square$

3.

Proposition 5. *The set*

$$\left\{n \in \mathbb{N} \mid n^4 \leq 3^n\right\} = \{1\} \cup \{n \in \mathbb{N} \mid n \geq 8\}$$

Proof. Denote the set $\{n \in \mathbb{N} \mid n^4 \leq 3^n\}$ by A. It is easy to check that $1,8 \in A$, but $2,\ldots,7 \notin A$. Suppose that $k \in A$ for some $k \geq 8$. Then due to the lemma below,

$$\begin{aligned}(k+1)^4 &= k^4+4k^3+6k^2+4k+1\\ &\leq k^4+2k^4\\ &= 3k^4\\ &\leq 3\cdot 3^k\\ &= 3^{k+1}.\end{aligned}$$

Thus $k+1 \in A$. $\square$

Lemma 6. *For every positive integer* $k \geq 8$,

$$2k^4 \geq 4k^3 + 6k^2 + 4k + 1.$$

Proof. For $k = 8$, the inequality holds since

$$2 \cdot 8^4 = 8192 \geq 2465 = 4 \cdot 8^3 + 6 \cdot 8^2 + 4 \cdot 8 + 1.$$

Now suppose that for some $k \geq 8$,

$$2k^4 \geq 4k^3 + 6k^2 + 4k + 1.$$

Since $k \geq 8$, we have that $3k^2 \geq 24k$, and $2k^4 \geq 2 \cdot 8^4 = 8192$. Hence

$$\begin{aligned} 2(k+1)^4 &= 2k^4 + 8k^3 + 12k^2 + 8k + 2 \\ &= 8k^3 + 9k^2 + 8k + 2 + (3k^2 + 2k^4) \\ &\geq 8k^3 + 9k^2 + 8k + 2 + (24k + 8192) \\ &> 4k^3 + 9k^2 + 28k + 15 \\ &= 4(k+1)^3 + 6(k+1)^2 + 4(k+1) + 1. \end{aligned}$$

The result now follows by induction. □

4. (a)

Proposition 7. *For every integer* $n \geq 5$,

$$\frac{8}{5} \leq \frac{F_{n+1}}{F_n} \leq \frac{13}{8}.$$

Proof. We have that $F_5 = 5$ and $F_6 = 8$, and so

$$\frac{F_6}{F_5} = \frac{8}{5}.$$

Now suppose that for some positive integer $k \geq 5$,

$$\frac{8}{5} \leq \frac{F_{k+1}}{F_k} \leq \frac{13}{8}.$$

Then reciprocating, we have

$$\frac{5}{8} \geq \frac{F_k}{F_{k+1}} \geq \frac{8}{13}.$$

Thus

$$\frac{13}{8} \geq 1 + \frac{F_k}{F_{k+1}} \geq \frac{21}{13}.$$

Notice that

$$\frac{F_{k+2}}{F_{k+1}} = \frac{F_{k+1} + F_k}{F_{k+1}} = 1 + \frac{F_k}{F_{k+1}}.$$

Thus we see that

$$\frac{8}{5} < \frac{21}{13} \leq \frac{F_{k+2}}{F_{k+1}} \leq \frac{13}{8}.$$

The result now follows by induction. □

(b)

Proposition 8. *For every integer $n \geq 1$,*

$$F_{n+1}^2 - F_nF_{n+2} = (-1)^n.$$

Proof. Since $F_1 = F_2 = 1$ and $F_3 = 2$, we see that

$$F_2^2 - F_1F_3 = -1 = (-1)^1.$$

Suppose that for some $n \geq 1$,

$$F_{n+1}^2 - F_nF_{n+2} = (-1)^n.$$

Then

$$\begin{aligned} F_{n+2}^2 - F_{n+1}F_{n+3} &= F_{n+2}(F_{n+1} + F_n) - F_{n+1}(F_{n+2} + F_{n+1}) \\ &= F_{n+2}F_{n+1} + F_{n+2}F_n - F_{n+1}F_{n+2} - F_{n+1}^2 \\ &= -(F_{n+1}^2 - F_{n+2}F_n) = -(-1)^n = (-1)^{n+1}. \end{aligned}$$

The result now follows by induction. □

(c)

Proposition 9. *For every $n \geq 1$,*

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

Proof. Before we begin the induction, we pause to point out that

$$\left(\frac{1+\sqrt{5}}{2}\right)^2 = 1 + \frac{1+\sqrt{5}}{2}$$

and

$$\left(\frac{1-\sqrt{5}}{2}\right)^2 = 1 + \frac{1-\sqrt{5}}{2}.$$

We will use these facts below.

For $n = 1$ the formula holds, since

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)\right] = \frac{1}{\sqrt{5}}\left[\frac{2\sqrt{5}}{2}\right] = 1 = F_1.$$

For $n = 2$ the formula holds since

$$\begin{aligned} &\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right] \\ &= \frac{1}{\sqrt{5}}\left[\left(1 + \frac{1+\sqrt{5}}{2}\right) - \left(1 + \frac{1-\sqrt{5}}{2}\right)\right] \\ &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)\right] = 1 = F_2. \end{aligned}$$

Now suppose that for some positive integer k the formula holds for all $n = 1, \ldots, k, k+1$. Then

$$\begin{aligned} F_{k+2} &= F_{k+1} + F_k \\ &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right] \\ &\qquad + \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k} - \left(\frac{1-\sqrt{5}}{2}\right)^{k}\right] \\ &= \frac{1}{\sqrt{5}}\left[\left(1 + \frac{1+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{k} - \left(1 + \frac{1-\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{k}\right] \\ &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{2}\left(\frac{1+\sqrt{5}}{2}\right)^{k} - \left(\frac{1-\sqrt{5}}{2}\right)^{2}\left(\frac{1-\sqrt{5}}{2}\right)^{k}\right] \\ &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+2}\right]. \end{aligned}$$

The result now follows by induction. □

5. (a) It is easy to see that 41 divides the integer $41^2 + 41 + 41$, so $n^2 + n + 41$ is not prime for $n = 41$.

   (b) The error is in the phrase "Since p_2 is in both groups." When $n = 1$, it is false that p_2 is in both of the groups $\{p_1, \ldots, p_n\}$ and $\{p_2, \ldots, p_{n+1}\}$ since p_2 is not in the first group.