Solutions to Homework Assignment 3
Math 74, Fall 2006

October 1, 2006

1. (a)

Proposition 1. *Let $f : X \to Y$. Then f is surjective if and only if*

$$f\left(f^{-1}[B]\right) = B$$

for every subset $B \subseteq Y$.

Proof. First, assume that f is surjective. Choose an arbitrary subset $B \subseteq Y$. We will show that $f(f^{-1}[B]) = B$. From Problem 7(b) on Homework Assignment 2, we have that $f(f^{-1}[B]) \subseteq B$. Hence it suffices to show that $B \subseteq f(f^{-1}[B])$. To that end, choose an arbitrary point $y \in B$. Since f is surjective, there exists $x \in X$ such that $y = f(x)$. Clearly

$$x \in \{z \mid f(z) \in B\} = f^{-1}[B].$$

Thus $y = f(x) \in f(f^{-1}[B])$. This demonstrates that $B \subseteq f(f^{-1}[B])$.
Conversely, assume that

$$f\left(f^{-1}[B]\right) = B$$

for every subset $B \subseteq Y$. Then

$$f\left(f^{-1}[Y]\right) = Y.$$

Since $X = f^{-1}[Y]$, we see that $Y = f(X)$, so f is surjective. □

(b)

Proposition 2. *Let $f : X \to Y$. Then f is injective if and only if*

$$f^{-1}[f(A)] = A$$

for every subset $A \subseteq X$.

Proof. Suppose that f is injective. We will show that

$$f^{-1}[f(A)] = A$$

for every subset $A \subseteq X$. Choose an arbitrary subset $A \subseteq X$. By Problem 7(a) on Homework Assignment 2, we have that $A \subseteq f^{-1}[f(A)]$. So if suffices to show that $f^{-1}[f(A)] \subseteq A$. To that end, choose a point $x \in f^{-1}[f(A)]$. Then $f(x) \in f(A)$. Thus there exists $\tilde{x} \in A$ such that $f(x) = f(\tilde{x})$. Since f is injective, $x = \tilde{x}$, and so $x \in A$. Thus we have demonstrated that $f^{-1}[f(A)] \subseteq A$.

Now suppose that

$$f^{-1}[f(A)] = A$$

for every subset $A \subseteq X$. We will show that f is injective. Choose points $x_1, x_2 \in X$ for which $f(x_1) = f(x_2)$. Then

$$\{x_1\} = f^{-1}[f(\{x_1\})] = f^{-1}[\{f(x_1)\}].$$

Since $x_2 \in f^{-1}[\{f(x_1)\}]$, it follows that $x_1 = x_2$. □

2. (a) This statement is false. Consider the map $f : \{1\} \to \{1, 2\}$ given by $1 \mapsto 2$. Then $f(f^{-1}[f(\varnothing)]) = \varnothing = f(\varnothing)$, and

$$f\left(f^{-1}[f(\{1\})]\right) = \{2\} = f(\{1\}).$$

However, f is not surjective since $1 \notin f(\{1\})$.

(b) This statement is false. Using the same function f as in part (a), we see that f satisfies

$$f\left(f^{-1}\left[f\left(f^{-1}[B]\right)\right]\right) = B,$$

for all subsets $B \subseteq \{1\}$. As was pointed out above, f is not surjective.

3. (a)

Proposition 3. *Suppose that $f : X \to Y$. Then f is injective if and only if*

$$f(A \cap B) = f(A) \cap f(B).$$

for all subsets $A, B \subseteq X$.

Proof. Assume that f is injective, and that A and B are subsets of X. We will show that $f(A \cap B) = f(A) \cap f(B)$. From the lecture notes as well as Problem 6(a) on Homework Assignment 2, we have that $f(A \cap B) \subseteq f(A) \cap f(B)$, so we only need to demonstrate that $f(A) \cap f(B) \subseteq f(A \cap B)$. To that end, choose a point $y \in f(A) \cap f(B)$. This means that y belongs to both $f(A)$ and $f(B)$. Since $y \in f(A)$, we may find a point $x_1 \in A$ such that $y = f(x_1)$. Likewise, since $y \in f(B)$, we may select a point $x_2 \in B$ for which $y = f(x_2)$. Since f is injective, and $f(x_1) = y = f(x_2)$, we deduce that $x_1 = x_2$. Thus $x_1 \in A \cap B$, and since $y = f(x_1)$, we see that $y \in f(A \cap B)$.

Now suppose that $f(A \cap B) = f(A) \cap f(B)$ for all subsets $A, B \subseteq X$. We must show that f is injective. To that end, select points $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$. Define $A = \{x_1\}$ and $B = \{x_2\}$. Then using our assumption, we have that

$$f(A \cap B) = f(A) \cap f(B) = \{f(x_1)\} \cap \{f(x_2)\} = \{f(x_1)\}.$$

This implies that $A \cap B$ is nonempty, for otherwise the set $f(A \cap B)$ would be empty. But $\{x_1\} \cap \{x_2\} \neq \varnothing$ implies that $x_1 = x_2$. □

(b) This statement if false. We will demonstrate this by proving that for any function $f : X \to Y$ and all subsets $C, D \subseteq Y$,

$$f^{-1}[C \cup D] = f^{-1}[C] \cup f^{-1}[D].$$

Not all functions are surjective (for example, $f : \{1\} \to \{1, 2\}$ given by $1 \mapsto 1$), and hence the statement is false.

Proposition 4. *For any function $f : X \to Y$ and for all subsets $C, D \subseteq Y$,*

$$f^{-1}[C \cup D] = f^{-1}[C] \cup f^{-1}[D].$$

Proof. Pick arbitrary subsets $C, D \subseteq Y$. Pick an arbitrary element $x \in f^{-1}[C \cup D]$. Then $f(x) \in C \cup D$. Thus $f(x) \in C$ or $f(x) \in D$. Thus $x \in f^{-1}[C]$ or $x \in f^{-1}[D]$. Thus $x \in f^{-1}[C] \cup f^{-1}[D]$. It follows that

$$f^{-1}[C \cup D] \subseteq f^{-1}[C] \cup f^{-1}[D].$$

On the other hand, it is obvious that $f^{-1}[C] \subseteq f^{-1}[C \cup D]$ and $f^{-1}[D] \subseteq f^{-1}[C \cup D]$. Thus

$$f^{-1}[C] \cup f^{-1}[D] \subseteq f^{-1}[C \cup D].$$

□

4. (a)

Proposition 5. *For every positive integer n,*

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$

Proof. Define a set A by

$$A = \left\{ n \in \mathbb{N}^* \mid \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4} \right\}.$$

Clearly $1 \in A$ since

$$1 = \sum_{k=1}^{n} k^3 = 1^3 = 1 = \frac{1^2(1+1)^2}{4}.$$

Now suppose that $m \in A$. Then m is a positive integer such that

$$\sum_{k=1}^{m} k^3 = \frac{m^2(m+1)^2}{4}.$$

Then

$$\begin{aligned}\sum_{k=1}^{m+1} k^3 &= (m+1)^3 + \sum_{k=1}^{m} k^3 \\ &= (m+1)^3 + \frac{m^2(m+1)^2}{4} \\ &= \frac{4(m+1)(m+1)^2 + m^2(m+1)^2}{4} \\ &= \frac{(m+1)^2(4m+4+m^2)}{4} \\ &= \frac{(m+1)^2(m+2)^2}{4}.\end{aligned}$$

Hence $m+1 \in A$. By induction, it follows that $A = \mathbb{N}^*$. □

(b)

Proposition 6. *For every positive integer n,*

$$\sum_{k=1}^{n} k^3 = \left[\sum_{k=1}^{n} k\right]^2.$$

Proof. In the lecture notes (and in class) we saw that

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

Thus from part (a) we have

$$\left[\sum_{k=1}^{n} k\right]^2 = \left[\frac{n(n+1)}{2}\right]^2 = \frac{n^2(n+1)^2}{4} = \sum_{k=1}^{n} k^3.$$

□

5. (a)

Proposition 7. *For every integer n,*

$$\sum_{k=1}^{n} \frac{1}{2^k} = 1 - \frac{1}{2^n}.$$

Proof. The statement is true for $n = 1$, since

$$\sum_{k=1}^{1} \frac{1}{2^k} = \frac{1}{2} = 1 - \frac{1}{2^1}.$$

Suppose that the statement is true for $n = m$; that is,

$$\sum_{k=1}^{m} \frac{1}{2^k} = 1 - \frac{1}{2^m}.$$

Then

$$\begin{aligned}
\sum_{k=1}^{m+1} \frac{1}{2^k} &= \frac{1}{2^{m+1}} + \sum_{k=1}^{m} \frac{1}{2^k} \\
&= \frac{1}{2^{m+1}} + \left(1 - \frac{1}{2^m}\right) \\
&= 1 + \frac{1}{2^{m+1}} - \frac{2}{2^{m+1}} \\
&= 1 - \frac{1}{2^{m+1}}.
\end{aligned}$$

Thus the statement is true for $n = m+1$, and the result now follows by induction. □

(b)

Proposition 8. *For every integer n,*

$$\sum_{k=1}^{n} \frac{1}{3^k} = \frac{1}{2}\left(1 - \frac{1}{3^n}\right).$$

Proof. The statement is true for $n = 1$, since

$$\sum_{k=1}^{1} \frac{1}{3^k} = \frac{1}{3} = \frac{1}{2}\left(1 - \frac{1}{3^1}\right).$$

Suppose that the statement is true for $n = m$; that is,

$$\sum_{k=1}^{m} \frac{1}{3^k} = \frac{1}{2}\left(1 - \frac{1}{3^m}\right).$$

Then

$$\begin{aligned}
\sum_{k=1}^{m+1} \frac{1}{3^k} &= \frac{1}{3^{m+1}} + \sum_{k=1}^{m} \frac{1}{3^k} \\
&= \frac{1}{3^{m+1}} + \frac{1}{2}\left(1 - \frac{1}{3^m}\right) \\
&= \frac{1}{2} + \frac{2}{2 \cdot 3^{m+1}} - \frac{3}{2 \cdot 3^{m+1}} \\
&= \frac{1}{2}\left(1 - \frac{1}{3^{m+1}}\right).
\end{aligned}$$

Thus the statement is true for $n = m+1$, and the result now follows by induction. $\square$

6. We will find a formula for the sum

$$\sum_{k=1}^{n} k^4$$

and then prove our formula is true by induction. To find a formula, we must first make a "flexible" guess. It is clear that

$$n^4 \leq \sum_{k=1}^{n} k^4 \leq n^5.$$

Moreover, we already saw that

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2},$$
$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

and so it seems that a good guess is

$$\sum_{k=1}^{n} k^4 = a_0 n^5 + a_1 n^4 + a_2 n^3 + a_3 n^2 + a_4 n + a_5.$$

We will now calculate the coefficients a_j under the assumption that our guess is indeed correct, and that we can prove our formula works by induction. First of all, we require that our formula works for $n = 1$, which means that

$$1 = \sum_{k=1}^{1} k^4 = a_0 + a_1 + a_2 + a_3 + a_4 + a_5.$$

Now if a proof by induction will work, it must be the case that assuming that our formula works for $n = m$ implies that it also holds true to $n = m + 1$. Let's sketch out how this proof would work:

Assume that

$$\sum_{k=1}^{m} k^4 = a_0 m^5 + a_1 m^4 + a_2 m^3 + a_3 m^2 + a_4 m + a_5.$$

Then

$$\begin{aligned}
\sum_{k=1}^{m+1} k^4 &= (m+1)^4 + \sum_{k=1}^{m} k^4 \\
&= (m^4 + 4m^3 + 6m^2 + 4m + 1) + a_0 m^5 + a_1 m^4 + a_2 m^3 + a_3 m^2 + a_4 m + a_5 \\
&= a_0 m^5 + (1 + a_1)m^4 + (4 + a_2)m^3 + (6 + a_3)m^2 + (4 + a_4)m + (1 + a_5).
\end{aligned}$$

Now expand

$$\begin{aligned}
&a_0(m+1)^5 + a_1(m+1)^4 + a_2(m+1)^3 + a_3(m+1)^2 + a_4(m+1) + a_5 \\
&= a_0 m^5 + (5a_0 + a_1)m^4 + (10a_0 + 4a_1 + a_2)m^3 + (10a_0 + 6a_1 + 3a_2 + a_3)m^2 \\
&\quad +(5a_0 + 4a_1 + 3a_2 + 2a_3 + a_4)m + (a_0 + a_1 + a_2 + a_3 + a_4 + a_5)
\end{aligned}$$

We will try to choose the coefficients a_j so that our two expressions are equal. Thus we require

$$1 + a_1 = 5a_0 + a_1 \qquad \Longrightarrow \qquad a_0 = \frac{1}{5},$$

and

$$4 + a_2 = 10a_0 + 4a_1 + a_2 \qquad \Longrightarrow \qquad a_1 = \frac{1}{2},$$

and

$$6 + a_3 = 10a_0 + 6a_1 + 3a_2 + a_3 \qquad \Longrightarrow \qquad a_2 = \frac{1}{3},$$

and

$$4 + a_4 = 5a_0 + 4a_1 + 3a_2 + 2a_3 + a_4 \qquad \Longrightarrow \qquad a_3 = 0,$$

and

$$1 + a_5 = a_0 + a_1 + a_2 + a_3 + a_4 + a_5 \qquad \Longrightarrow \qquad a_4 = -\frac{1}{30}.$$

Remembering that $1 = \sum_{k=1}^{1} k^4 = a_0 + a_1 + a_2 + a_3 + a_4 + a_5$, we see that $a_5 = 0$. Hence our guess is that

$$\sum_{k=1}^{n} k^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}.$$

Proposition 9. *For every positive integer n,*

$$\sum_{k=1}^{n} k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}.$$

Proof. Clearly for formula above is correct for $n = 1$, since in this case it reads

$$\sum_{k=1}^{1} k^4 = 1 = \frac{1(1+1)(2+1)(3+3-1)}{30}.$$

Now assume that for some positive integer m, the formula is true for $n = m$. Then we calculate

$$\begin{aligned}
\sum_{k=1}^{m+1} k^4 &= (m+1)^4 + \sum_{k=1}^{m} k^4 \\
&= (m+1)^4 + \frac{m(m+1)(2m+1)(3m^2+3m-1)}{30}.
\end{aligned}$$

An easy calculation demonstrates that

$$\begin{aligned}(m+1)^4 &+ \frac{m(m+1)(2m+1)(3m^2+3m-1)}{30} \\ &= \frac{(m+1)(m+2)(2m+3)(3(m+1)^2+3(m+1)-1)}{30}.\end{aligned}$$

Thus our formula is true for $n = m + 1$, and the result now follows by induction. □

7. (a) This statement is false. Consider the functions $f : \{1\} \to \{1, 2\}$ and $g : \{1\} \to \{1, 2\}$ given by $f(1) = 1$ and $g(1) = 2$. Then

$$f\left(g^{-1}[\{1,2\}]\right) = f(\{1\}) = \{1\},$$

but

$$g\left(f^{-1}[\{1,2\}]\right) = g(\{1\}) = \{2\},$$

and so $f\left(g^{-1}[\{1,2\}]\right) \neq g\left(f^{-1}[\{1,2\}]\right)$.

(b) This statement is false. Consider the functions $f : \{1, 2\} \to \{1, 2\}$ and $g : \{1, 2\} \to \{1, 2\}$ given by $f(x) = x$ and $g(x) = 2$ for each $x \in \{1, 2\}$. Then

$$f^{-1}[g(\{1,2\})] = f^{-1}[\{2\}] = \{2\}.$$

However,

$$g^{-1}[f(\{1,2\})] = g^{-1}[\{1,2\}] = \{1,2\}.$$

Thus $f^{-1}[g(\{1,2\})] \neq g^{-1}[f(\{1,2\})]$.