Solutions to Homework Assignment 2
Math 74, Fall 2006

October 1, 2006

1. (a) If P and Q are statements, then using De Morgan's laws we see that

$$\neg\,[\neg P \wedge \neg Q] \Leftrightarrow \neg\neg P \vee \neg\neg Q \Leftrightarrow P \vee Q.$$

Likewise,

$$\neg\,[P \wedge \neg Q] \Leftrightarrow P \Rightarrow Q,$$

and

$$(\neg\,[P \wedge \neg Q]) \wedge (\neg\,[Q \wedge \neg P]) \Leftrightarrow (P \Leftrightarrow Q).$$

So we see that we can build all of the logical connectives out of $\neg$ and $\wedge$.

(b) Let P, Q and R be statements. Then using our work above we have

$$\begin{aligned}
&[(P \Rightarrow Q) \vee R] \Rightarrow [\neg Q \vee (R \vee P)] \\
&\Leftrightarrow \quad \neg\left([(P \Rightarrow Q) \vee R] \wedge \neg\,[\neg Q \vee (R \vee P)]\right) \\
&\Leftrightarrow \quad \neg\left([\neg(P \wedge \neg Q) \vee R] \wedge [Q \wedge \neg(R \vee P)]\right) \\
&\Leftrightarrow \quad \neg\left(\neg\,[\neg R \wedge P \wedge \neg Q] \wedge [Q \wedge \neg R \wedge \neg P]\right).
\end{aligned}$$

From this point, it is easy to see that the statement is equivalent to $\neg\,[Q \wedge \neg R \wedge \neg P]$.

2. (a) We will show that we can write the five common logical connectives $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$ by using only the logical connective$\uparrow$. It is clear that

$$P \uparrow P \Leftrightarrow \neg\,[P \wedge P] \Leftrightarrow \neg P.$$

Using De Morgan's laws, we see that

$$(P \uparrow P) \uparrow (Q \uparrow Q) \Leftrightarrow (\neg P) \uparrow (\neg Q) \Leftrightarrow \neg\,[\neg P \wedge \neg Q] \Leftrightarrow P \vee Q.$$

Also,

$$(P \uparrow Q) \uparrow (P \uparrow Q) \Leftrightarrow \neg\,(P \uparrow Q) \Leftrightarrow \neg\,[\neg(P \wedge Q)] \Leftrightarrow P \wedge Q.$$

Also,

$$\begin{aligned}
[(P \uparrow P) \uparrow (P \uparrow P)] \uparrow [Q \uparrow Q] \quad &\Leftrightarrow \quad [(\neg P) \uparrow (\neg P)] \uparrow [Q \uparrow Q] \\
&\Leftrightarrow \quad (\neg P) \vee Q \\
&\Leftrightarrow \quad (P \Rightarrow Q).
\end{aligned}$$

From our work above we have that

$$\begin{aligned}[P \Leftrightarrow Q] &\Leftrightarrow [P \Rightarrow Q] \wedge [Q \Rightarrow P] \\ &\Leftrightarrow [(P \Rightarrow Q) \uparrow (Q \Rightarrow P)] \uparrow [(P \Rightarrow Q) \uparrow (Q \Rightarrow P)] \\ &\Leftrightarrow [R \uparrow S] \uparrow [R \uparrow S],\end{aligned}$$

where the symbol R is used in place of the statement

$$[(P \uparrow P) \uparrow (P \uparrow P)] \uparrow [Q \uparrow Q]$$

and S is used in place of

$$[(Q \uparrow Q) \uparrow (Q \uparrow Q)] \uparrow [P \uparrow P].$$

(b) We will show that we can write the five common logical connectives $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$ by using only the logical connective$\downarrow$. It is clear that

$$P \downarrow P \Leftrightarrow \neg [P \vee P] \Leftrightarrow \neg P.$$

Using De Morgan's laws,

$$(P \downarrow P) \downarrow (Q \downarrow Q) \Leftrightarrow \neg [(\neg P) \vee (\neg Q)] \Leftrightarrow P \wedge Q.$$

Also,

$$(P \downarrow Q) \downarrow (P \downarrow Q) \Leftrightarrow \neg (P \downarrow Q) \Leftrightarrow \neg [\neg (P \vee Q)] \Leftrightarrow P \vee Q.$$

Also,

$$\begin{aligned}[(P \downarrow P) \downarrow (P \downarrow P)] \downarrow [Q \downarrow Q] &\Leftrightarrow [(\neg P) \downarrow (\neg P)] \downarrow [Q \downarrow Q] \\ &\Leftrightarrow (\neg P) \vee Q \\ &\Leftrightarrow (P \Rightarrow Q).\end{aligned}$$

From our work above we have that

$$\begin{aligned}[P \Leftrightarrow Q] &\Leftrightarrow [P \Rightarrow Q] \wedge [Q \Rightarrow P] \\ &\Leftrightarrow [R \downarrow R] \downarrow [S \downarrow S],\end{aligned}$$

where the symbol R is used in place of the statement

$$[(P \downarrow P) \downarrow (P \downarrow P)] \downarrow [Q \downarrow Q]$$

and S is used in place of

$$[(Q \downarrow Q) \downarrow (Q \downarrow Q)] \downarrow [P \downarrow P].$$

(c) From part (a), we see that

$$\begin{aligned}(P \Rightarrow \neg Q) \wedge R &\Leftrightarrow [(P \Rightarrow \neg Q) \uparrow R] \uparrow [(P \Rightarrow \neg Q) \uparrow R] \\ &\Leftrightarrow [S \uparrow R] \uparrow [S \uparrow R]\end{aligned}$$

where S stands for the statement

$$[(P \uparrow P) \uparrow (P \uparrow P)] \uparrow [(Q \uparrow Q) \uparrow (Q \uparrow Q)] .$$

From part (b), we see that

$$\begin{aligned}(P \Rightarrow \neg Q) \wedge R &\Leftrightarrow [(P \Rightarrow \neg Q) \downarrow (P \Rightarrow \neg Q)] \downarrow [R \downarrow R] \\ &\Leftrightarrow [T \downarrow T] \downarrow [R \downarrow R]\end{aligned}$$

where T stands for the statement

$$[(P \downarrow P) \downarrow (P \downarrow P)] \downarrow [(Q \downarrow Q) \downarrow (Q \downarrow Q)] .$$

3. (a) This statement is false. Consider the sets $A = \{1, 2\}$, $B = \{1\}$, and $C = \{2\}$. Then clearly $C \cap (A \cap B) = \varnothing$, but $C \cap A = \{2\}$ and $B \cap A = \{1\}$.

(b)

Proposition 1. *Suppose that A, B, and C are sets such that C is a subset of both A and B. Then $C \subseteq A \cap B$.*

Proof. Pick an element $x \in C$. Then $x \in A$ since $C \subseteq A$, and $x \in B$ since $C \subseteq B$. Thus $x \in A \cap B$. □

(c) This statement is false. Consider the same sets we used in part (a): $A = \{1, 2\}$, $B = \{1\}$, and $C = \{2\}$. Then $A \backslash B = \{2\}$, and so $(A \backslash B) \backslash C = \varnothing$. Also notice that $B \backslash C = \{1\}$, so $A \backslash (B \backslash C) = \{2\}$. Thus $(A \backslash B) \backslash C \neq A \backslash (B \backslash C)$.

(d)

Proposition 2. *Suppose that A, B, and C are sets such that $(A \cap B) \cup C \subseteq A \cap (B \cup C)$. Then $C \subseteq A$.*

Proof. Let $x \in C$. Then $x \in (A \cap B) \cup C$, which implies that $x \in A \cap (B \cup C)$. Thus $x \in A$. □

4. (a)

Proposition 3. *Let A, B, and X be sets such that $A \cup B \subseteq X$. Then*

$$A \backslash B = (X \backslash B) \backslash (X \backslash A).$$

Proof. Pick a point $x \in A \backslash B$. Then $x \in A$ and $x \notin B$. Since $A \subseteq X$, this implies that $x \in X$. Thus $x \in X \backslash B$ and $x \notin X \backslash A$, which implies that $x \in (X \backslash B) \backslash (X \backslash A)$.

Now suppose that $y \in (X \backslash B) \backslash (X \backslash A)$. Then $y \in (X \backslash B)$ but $y \notin (X \backslash A)$. So we see that $y \in X$, but $y \notin B$. Also, it must be that $y \in A$, for otherwise we would have $y \in (X \backslash A)$. Thus y is an element of A but not B, so $y \in A \backslash B$. $\square$

(b) This statement is false. Consider the (only) function $f : \{1, 2\} \to \{1\}$, and let $A = \{1, 2\}$ and $B = \{1\}$. Then $f(A) = f(B) = \{1\}$, so $f(A) \backslash f(B) = \varnothing$. However, $f(A \backslash B) = f(\{2\}) = \{1\}$. Thus $f(A) \backslash f(B) \neq f(A \backslash B)$.

(c)

Proposition 4. *Let $f : X \to Y$ be a function, and suppose that $C, D \subseteq Y$. Then*

$$f^{-1}[C] \backslash f^{-1}[D] = f^{-1}[C \backslash D].$$

Proof. Pick an element $x \in f^{-1}[C] \backslash f^{-1}[D]$. This means that $x \in f^{-1}[C]$ and $x \notin f^{-1}[D]$. Hence $f(x) \in C$ but $f(x) \notin D$. This implies that $f(x) \in C \backslash D$, and hence $x \in f^{-1}[C \backslash D]$.

Now pick an element $z \in f^{-1}[C \backslash D]$. Then $f(z) \in C \backslash D$. Hence $f(z) \in C$ but $f(z) \notin D$. Thus $z \in f^{-1}[C]$ but $z \notin f^{-1}[D]$, and so $z \in f^{-1}[C] \backslash f^{-1}[D]$. $\square$

5. (a)

Proposition 5. *Suppose that X and Y are sets, and $x \in X$ and $y \in Y$. Then the ordered pair $(x, y) \subseteq \mathcal{P}(X \cup Y)$.*

Proof. Let $z \in (x, y) = \{\{x\}, \{x, y\}\}$. Then either $z = \{x\}$ or $z = \{x, y\}$. In the first case, clearly $z \subseteq X \subseteq X \cup Y$ since $x \in X$.. In the second case, $z \subseteq X \cup Y$, since $x \in X$ and $y \in Y$. Either way, $z \subseteq X \cup Y$. Hence $z \in \mathcal{P}(X \cup Y)$. $\square$

(b)

Proposition 6. *If X and Y are sets, then the Cartesian product $X \times Y \in \mathcal{P}(\mathcal{P}(\mathcal{P}(X \cup Y)))$.*

Proof. Let $(x, y) \in X \times Y$. Then from the proposition above, $(x, y) \subseteq \mathcal{P}(X \cup Y)$. Hence $(x, y) \in \mathcal{P}(\mathcal{P}(X \cup Y))$. Therefore,

$$X \times Y \subseteq \mathcal{P}(\mathcal{P}(X \cup Y)).$$

Thus

$$X \times Y \in \mathcal{P}(\mathcal{P}(\mathcal{P}(X \cup Y))).$$

$\square$

6. (a)

Proposition 7. *Let $f : X \to Y$ be a map, and suppose that $A, B \subseteq X$. Then*

$$f(A \cap B) \subseteq f(A) \cap f(B).$$

Proof. Let $y \in f(A \cap B)$. Then there exists an element $x \in A \cap B$ such that $f(x) = y$. Since $x \in A$, and $y = f(x)$, we see that $y \in f(A)$. Likewise, $y \in f(B)$. Thus $y \in f(A) \cap f(B)$. □

Equality in the above proposition need not hold. For example, consider the function $f : \{1, 2\} \to \{1\}$, and sets $A = \{1\}$ and $B = \{2\}$. Then $A \cap B = \varnothing$, and so $f(A \cap B) = \varnothing$ while $f(A) \cap f(B) = \{1\}$.

(b)

Proposition 8. *Let $f : X \to Y$ be a map, and suppose that $A, B \subseteq X$. Then*

$$f(A \cup B) = f(A) \cup f(B).$$

Proof. Pick an element $y \in f(A \cup B)$. Then there exists an element $x \in A \cup B$ such that $y = f(x)$. If $x \in A$, then clearly $y \in f(A)$. Likewise, if $x \in B$ then $y \in f(B)$. Either way, $y \in f(A) \cup f(B)$.

Conversely, suppose that y is an element of the set $f(A) \cup f(B)$. If $y \in f(A)$, then there exists an element $x \in A \subseteq A \cup B$ such that $y = f(x)$. If $y \in f(B)$, then there exists an element $z \in B \subseteq A \cup B$ such that $y = f(z)$. Either way, it is clear that $y \in f(A \cup B)$. □

(c)

Proposition 9. *Suppose that $f : X \to Y$ is a function and A and B are subsets of X. Then*

$$f(A) \vartriangle f(B) \subseteq f(A \vartriangle B).$$

Proof. Suppose that y is an element of $f(A) \vartriangle f(B)$. Then $y \in f(A)$ or $y \in f(B)$, but not both. In the first case, that $y \in f(A)$, there exists an element $x \in A$ such that $y = f(x)$. Since $y \notin f(B)$, it must be the case that $x \notin B$. Hence $x \in A \vartriangle B$, and we deduce that $y \in f(A \vartriangle B)$. If, on the other hand, $y \in f(B)$, then there exists an element $z \in B$ such that $y = f(z)$. Since $y \notin f(A)$, we deduce that $z \in A \vartriangle B$ and $y \in f(A \vartriangle B)$. In both cases, $y \in f(A \vartriangle B)$. □

Equality need not hold in the above proposition. For example, suppose that $f : \{1, 2\} \to \{1\}$, let $A = \{1\}$ and $B = \{2\}$. Then $A \vartriangle B = \{1, 2\}$, and hence $f(A \vartriangle B) = \{1\}$. However, $f(A) = f(B) = \{1\}$, so $f(A) \vartriangle f(B) = \varnothing$. Thus

$$f(A) \vartriangle f(B) \neq f(A \vartriangle B).$$

7. (a)

Proposition 10. *Let $f : X \to Y$ be a function, and suppose that $A \subseteq X$. Then*

$$A \subseteq f^{-1}\left[f\left(A\right)\right].$$

Proof. Pick an arbitrary element $x \in A$. Then clearly $f(x) \in \{f(z) \mid z \in A\} = f(A)$. Thus $x \in \{z \in X \mid f(z) \in f(A)\} = f^{-1}[f(A)]$. □

Equality need not hold in the above proposition. Consider the function $f : \{1,2\} \to \{1\}$, and $A = \{1\}$. Then $f^{-1}[f(A)] = \{1,2\} \neq A$.

(b)

Proposition 11. *Let $f : X \to Y$ be a function, and suppose that $B \subseteq Y$. Then*

$$f\left(f^{-1}[B]\right) \subseteq B.$$

Proof. Let y be an arbitrary element of the set

$$f\left(f^{-1}[B]\right) = \left\{f(x) \in Y \mid x \in f^{-1}[B]\right\}.$$

Then there exists an $x \in f^{-1}[B]$ such that $y = f(x)$. But since $x \in f^{-1}[B]$, we see that $f(x) \in B$. Thus $y \in B$. □

Equality need not hold in the above proposition. Consider the function $f : \{1\} \to \{1,2\}$ given by $1 \mapsto 2$, and set $B = \{1\}$. Then $f\left(f^{-1}[B]\right) = \varnothing \neq B$.

(c)

Proposition 12. *If $f : X \to Y$ and $B \subseteq Y$, then $f^{-1}[Y \backslash B] = X \backslash f^{-1}[B]$.*

Proof. Pick a point $x \in f^{-1}[Y \backslash B]$. This means that $f(x) \in Y \backslash B$. Hence $f(x) \notin B$. Thus $x \notin f^{-1}[B]$, and so $x \in X \backslash f^{-1}[B]$.

Now pick a point $z \in X \backslash f^{-1}[B]$. Then $f(z) \notin B$. Hence $f(z) \in Y \backslash B$, which implies that $z \in f^{-1}[Y \backslash B]$. □