Solutions to Homework Assignment 1
Math 74, Fall 2006

September 15, 2006

1. (a) The statement $\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q)$ can be read as "neither P nor Q if and only if not P and not Q." The statement $\neg(P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q)$ can be read as "not both P and Q if and only if not P or not Q."

   (b) The following truth table establish that the two statements above are tautologies:

P	Q	$\neg(P \vee Q)$	$\neg P \wedge \neg Q$	$\neg(P \wedge Q)$	$\neg(P \vee Q)$
T	T	F	F	F	T
T	F	F	F	T	T
F	T	F	F	T	T
F	F	T	T	T	T

   (c)

   Proposition 1. *Let X be a set, and let A and B be subsets of X. Then*

$$X \backslash (A \cup B) = (X \backslash A) \cap (X \backslash B).$$

   Proof. Suppose that x is an element of $X \backslash (A \cup B)$. This means that $x \in X$ but $x \notin A \cup B$. That is, x is not an element of A or B. Thus x belongs to both of the sets $(X \backslash A)$ and $(X \backslash B)$. Therefore, $x \in (X \backslash A) \cap (X \backslash B)$. So we have shown that $X \backslash (A \cup B) \subseteq (X \backslash A) \cap (X \backslash B)$.

   On the other hand, if $y \in (X \backslash A) \cap (X \backslash B)$, then y belongs to both $X \backslash A$ and $X \backslash B$. Therefore, y belongs to X but is not a member of A or B. Thus, $y \notin A \cup B$, so $y \in X \backslash (A \cup B)$. Therefore, $(X \backslash A) \cap (X \backslash B) \subseteq X \backslash (A \cup B)$. □

   (d) The formula we obtained in the proposition above is related to the tautology

$$\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q).$$

   Suppose that $x \in X$. If we let P be the statement "$x \in A$" and Q be the statement "$x \in B$", then we see that the statement that x belongs to the set $X \backslash (A \cup B)$ is simply the statement $\neg(P \vee Q)$, while the statement that x belongs to the set $(X \backslash A) \cap (X \backslash B)$ is the statement $\neg P \wedge \neg Q$.

(e)

Proposition 2. *Let X be a set, and let A and B be subsets of X. Then*

$$X \backslash (A \cap B) = (X \backslash A) \cup (X \backslash B).$$

Proof. Suppose that x is an element of $X \backslash (A \cap B)$. Then $x \in X$ but $x \not\in (A \cap B)$. Thus x is not an element of both A and B. Therefore, x is an element of $X \backslash A$ or x is an element of $X \backslash B$. Either way, x is an element of the set $(X \backslash A) \cup (X \backslash B)$. So we see that $X \backslash (A \cap B) \subseteq (X \backslash A) \cup (X \backslash B)$.

On the other hand, if y is an element of $(X \backslash A) \cup (X \backslash B)$, then $y \in X$ and y is an element of $(X \backslash A)$ or $(X \backslash B)$. Since $y \in X$, we see that y cannot be an element of both A and B. This means that $y \notin A \cap B$. So $y \in X \backslash (A \cap B)$. Therefore, $(X \backslash A) \cup (X \backslash B) \subseteq X \backslash (A \cap B)$.

□

2. (a) The following truth table confirms that the statement

$$[(P \vee Q) \vee R] \Leftrightarrow [P \vee (Q \vee R)].$$

is a tautology.

(b) The following truth table establish that the two statements above are tautologies:

P	Q	R	$(P \vee Q)$	$(Q \vee R)$	$(P \vee Q) \vee R$	$P \vee (Q \vee R)$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
T	F	F	T	F	T	T
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	F	T	T	T
F	F	F	F	F	F	F

(c) The following truth table confirms that the statement

$$[(P \wedge Q) \wedge R] \Leftrightarrow [P \wedge (Q \wedge R)].$$

is a tautology.

P	Q	R	$(P \wedge Q)$	$(Q \wedge R)$	$(P \wedge Q) \wedge R$	$P \wedge (Q \wedge R)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	F	F	F
T	F	F	F	F	F	F
F	T	T	F	T	F	F
F	T	F	F	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

(d) The statement

$$[(P \wedge Q) \vee R] \Leftrightarrow [P \wedge (Q \vee R)]\,.$$

is not a tautology, since the left side is true but the right side is false in the case that P and Q are false but R is true.

(e) The logical connectives $\wedge$ and $\vee$ are binary operators– they operate on exactly two statements. However, due to part (a), we can interpret $P \vee Q \vee R$ unambiguously since it doesn't matter if we interpret it as $P \vee (Q \vee R)$ or $(P \vee Q) \vee R$, since these statements are equivalent. Likewise, owing to part (b) we can make sense of $P \wedge Q \wedge R$. However, as part (c) demonstrates, the notation $P \vee Q \wedge R$ is ambiguous, since we are not sure where to put the parentheses (and it matters)!

3. (a) We see that the statement

$$[P \Leftrightarrow Q] \Leftrightarrow [(P \wedge Q) \vee \neg(P \vee Q)]$$

is a tautology via the truth table below.

P	Q	$P \Leftrightarrow Q$	$P \wedge Q$	$\neg(P \vee Q)$	$(P \wedge Q) \vee \neg(P \vee Q)$
T	T	T	T	F	T
T	F	F	F	F	F
F	T	F	F	F	F
F	F	T	F	T	T

(b) Using De Morgan's laws, we see that

$$\begin{aligned} [(Q \wedge R) \Rightarrow (P \vee T)] &\Leftrightarrow [\neg(Q \wedge R) \vee (P \vee T)] \\ &\Leftrightarrow [(\neg Q \vee \neg R) \vee (P \vee T)] \\ &\Leftrightarrow [P \vee \neg Q \vee \neg R \vee T]\,. \end{aligned}$$

Therefore, the statement

$$[(Q \wedge R) \Rightarrow (P \vee T)] \Leftrightarrow [P \vee \neg Q \vee \neg R \vee T]$$

is a tautology.

(c) If P is false and Q and R are both true, then we see that the statement $\neg\,[P \Rightarrow \neg(Q \wedge R)]$ is false, while the statement $[P \vee Q \vee R]$ is true. Thus the statement

$$\neg\,[P \Rightarrow \neg(Q \wedge R)] \Leftrightarrow [P \vee Q \vee R]\,.$$

is not a tautology.

(d) If P and S are true, but Q and R are false, then the statement $[(P \wedge Q) \vee (R \wedge S)]$ is false, but the statement $[(P \vee R) \wedge (Q \vee S)]$ is true, so in this case the statement

$$[(P \wedge Q) \vee (R \wedge S)] \Leftrightarrow [(P \vee R) \wedge (Q \vee S)]$$

is false, and so it is not a tautology.

4. (a) There exists an integer n such that for all integers m, $m \neq 2n$.
   (b) For all integers n, there exists an integer m such that $m \neq 2n$.
   (c) There exists $y \in B$ such that for every $x \in A$, $f(x) \neq y$.
   (d) There exists $y \in \mathbb{R}$ and a positive number $\varepsilon > 0$ such that for all $\delta > 0$ there exists $y \in \mathbb{R}$ such that $|x - y| < \delta$ and $|f(x) - f(y)| \geq \varepsilon$.
   (e) For all $L > 0$, there exists $x, y \in \mathbb{R}$ such that $|f(x) - f(y)| \geq L$.
   (f) There exists $m, n, k \in \mathbb{N}$ such that $m^2 = n$ and $n = k^3$.

5. (a)

Proposition 3. *Suppose that X is a set and that $A, B \subseteq X$. Then $A \subseteq B$ if and only if A and $X \backslash B$ are disjoint.*

Proof. First, suppose that A and $X \backslash B$ are not disjoint. We will show that A is not a subset of B. Since we are assuming that $A \cap X \backslash B \neq \varnothing$, we may pick an element $x \in A \cap X \backslash B$. Then obviously $x \in A$, and $x \in X \backslash B$. From this it follows that $x \notin B$. Thus $x \in A$ but $x \notin B$, so A is not a subset of B.

Conversely, suppose that A is not a subset of B. Then we may select an element $y \in A$ for which $y \notin B$. Since A is a subset of X, it follows that y belongs to X. Thus $y \in X$ and $y \notin B$, so $y \in X \backslash B$. Thus $y \in A \cap (X \backslash B)$, so the set $A \cap (X \backslash B)$ is not empty. □

(b)

Proposition 4. *Suppose that A and B are subsets of a set X. Then $A = B$ if and only if $X \backslash A = X \backslash B$.*

Proof. Suppose that $A = B$. Pick an arbitrary element $x \in X \backslash A$. Then $x \in X$ but $x \notin A = B$. Then $x \in X \backslash B$. Therefore, $X \backslash A \subseteq X \backslash B$. By a similar argument, we see that $X \backslash B \subseteq X \backslash A$. Thus we see that $X \backslash A = X \backslash B$.

Now suppose that $A \neq B$. Then $A \nsubseteq B$ or $B \nsubseteq A$. Without loss of generality, suppose that $A \nsubseteq B$. Then there is an element $x \in A$ such that $x \notin B$. Since $A \subseteq X$, we see that $x \in X$ as well. Thus $x \in X$ but $x \notin B$, so $x \in X \backslash B$. But $x \notin X \backslash A$, since $x \in A$. Therefore, $X \backslash B \nsubseteq X \backslash A$. □

6. (a)

Proposition 5. *For all sets A and B,*

$$A \triangle B = A \cup B \backslash (A \cap B) = (A \backslash B) \cup (B \backslash A).$$

Proof. The set $A \cup B$ consists of those elements belonging to A or B, and the set $A \cap B$ is the set of those elements belonging to both A and B. Therefore, the set $A \cup B \backslash (A \cap B)$ consists of those elements belonging to A or B, but not both. Therefore, $A \triangle B = A \cup B \backslash (A \cap B)$.

The set $A\backslash B$ consists of those elements which belong to A but not B. Likewise, the set $B\backslash A$ consists of those elements which belong to B but not A. Thus the set $(A\backslash B) \cup (B\backslash A)$ consists of those elements which belong to A or B, but not both. Therefore, $A\triangle B = (A\backslash B) \cup (B\backslash A)$. □

(b)

Proposition 6. *For all sets A, B and C,*

$$A\triangle(B\triangle C) = (A\triangle B)\triangle C.$$

Proof. Suppose that $x \in A\triangle(B\triangle C)$. There are four possibilities, which we will handle one at a time.

Case 1: the element $x \in A$ and $x \in B$. Since $x \in A$, it must be that $x \notin B\triangle C$. Since $x \in B$, it must be that $x \in C$. Thus $x \in (A\triangle B)\triangle C$, since $x \in C$ but $x \notin A\triangle B$.

Case 2: the element $x \in A$ and $x \notin B$. Since $x \in A$, it must be that $x \notin B\triangle C$. Since $x \notin B$, it must be that $x \notin C$. Thus $x \in (A\triangle B)\triangle C$, since $x \notin C$ but $x \in A\triangle B$.

Case 3: the element $x \notin A$ and $x \in B$. Since $x \notin A$, it must be that $x \in B\triangle C$. Since $x \in B$, it must be that $x \notin C$. Thus $x \in (A\triangle B)\triangle C$, since $x \notin C$ but $x \in A\triangle B$.

Case 4: the element $x \notin A$ and $x \notin B$. Since $x \notin A$, it must be that $x \in B\triangle C$. Since $x \notin B$, it must be that $x \in C$. Thus $x \in (A\triangle B)\triangle C$, since $x \in C$ but $x \notin A\triangle B$.

In all four cases, $x \in x \in (A\triangle B)\triangle C$. So we have shown that

$$A\triangle(B\triangle C) \subseteq (A\triangle B)\triangle C.$$

For the other direction, repeat the argument above with the roles of A and C reversed. □