Homework Assignment 5

Due: Monday, November 6

1. **Definition.** If x is a real number, the **floor** of x is the greatest integer less than or equal to x. We denote the floor of x by $\lfloor x \rfloor$. If m and n are integers, then we say that m is a **multiple** of n and that n **divides** m and that n is a divisor of m if there exists an integer k for which $m = kn$.

   Suppose that $m, k \in \mathbb{Z}$ and $n \in \mathbb{N}^*$.

   (a) Show that $\left\lfloor \frac{m}{n} \right\rfloor = m \operatorname{div} n$.

   (b) Show that n divides m if and only if $m \bmod n = 0$.

   (c) Show that $(m \bmod n)(k \bmod n) \bmod n = (mk) \bmod n$.

   (d) Show that $(m \bmod n + k \bmod n) \bmod n = (m + k) \bmod n$.

2. Show that an integer is a multiple of 3 if and only if the sum of the digits in its base 10 representation is a multiple of 3; in other words, show that if

   $$a = \sum_{j=0}^{n} d_j 10^j$$

   then a is a multiple of 3 if and only if

   $$b = \sum_{j=0}^{n} d_j$$

   is a multiple of 3.

3. Let $m, n \in \mathbb{N}^*$.

   (a) Show that $\gcd(m, n) = \gcd(m + n, n)$.

   (b) Under what conditions is $\gcd(m, n) = \gcd(mn, n)$?

   (c) If k is a positive integer, show that $\gcd(km, kn) = k \cdot \gcd(m, n)$.

4. Let $m, n, k \in \mathbb{N}^*$.

   (a) Show that if p is a prime and $p|(mn)$ then either $p|m$ or $p|n$.

   (b) Suppose that m and n are relatively prime, and k and n are relatively prime. Show that km and n are relatively prime.

   (c) Show that m and n are relatively prime if and only if m^2 and n^2 are relatively prime.

5. Let m and n be positive integers.

(a) Come up with a good definition of a *common multiple* of m and n. Also define the *least common multiple* of m and n, and prove a lemma that shows that your definition makes sense. Denote the least common multiple of m and n by $\operatorname{lcm}(m, n)$.

(b) Show that $\operatorname{lcm}(m, n)$ is the unique common multiple of m and n that is a divisor of every common multiple of m and n. (Hint: think in an “ideal” way.)

(c) Show that $mn = \gcd(m, n) \cdot \operatorname{lcd}(m, n)$. (Hint: First show that $mn/\gcd(m, n)$ is an integer and a common multiple of m and n. Then show that it is the greatest one.)