Homework Assignment 4

Due: Wednesday, October 4

1. Invent a good definition for the term *perfect square*. Then use your definition and the Well-Ordering Principle to show that if $n \in \mathbb{N}$ then $\sqrt{n} \in \mathbb{Q}$ if and only if n is a perfect square (recall that $\mathbb{Q}$ is the set of rational numbers).

2. (a) Calculate the set
$$\mathcal{F} = \{5m + 7n \,|\, m, n \in \mathbb{N}\}.$$
   (b) For any $n \geq 32$ it is possible to fill an order for n pounds of fish using bottomless wheelbarrows full of five-pound and nine-pound fish. Rephrase this fact as a mathematical proposition, and then prove it.

3. Calculate the set
$$A = \left\{n \in \mathbb{N}^* \mid n^4 \leq 3^n\right\}$$

4. Recall that the **Fibonacci sequence** is defined recursively by
$$\begin{cases} F_1 = F_2 = 1, \\ F_{n+2} = F_{n+1} + F_n. \end{cases}$$
   (a) Show that for every integer $n \geq 5$,
$$\frac{8}{5} \leq \frac{F_{n+1}}{F_n} \leq \frac{13}{8}.$$
   (b) Show that for every integer $n \geq 1$,
$$F_{n+1}^2 - F_n F_{n+2} = (-1)^n$$
   (c) Show that for every $n \geq 1$,
$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

5. (a) Prove the following statement, or give a counterexample: for every $n \in \mathbb{N}^*$, the integer $n^2 + n + 41$ is prime.

   (b) Criticize the following incorrect "proof" (you are not required to prove anything, just explain clearly what are the flaws in the following "proof").

   We will show that in any group of n people, everyone is the same age. The proposition is obvious if $n = 1$. So suppose that the proposition is true for all groups consisting of n people. Let $G = \{p_1, p_2, ..., p_{n+1}\}$ be any group of $n+1$ distinct people. Since each of the groups $\{p_1, p_2, \ldots, p_n\}$ and $\{p_2, p_3, \ldots, p_{n+1}\}$ consists of n people, everybody in each of these groups has the same age, by our induction hypothesis. Since p_2 is in both groups, it follows that every person belonging to G has the same age as p_2. Thus the proposition is true for $n+1$. By induction, the proof is complete.