

Homework Assignment 3

Due: Wednesday, September 27

1. Let $f : X \rightarrow Y$.

(a) Show that f is surjective if and only if

$$f(f^{-1}[B]) = B$$

for all subsets $B \subseteq Y$.

(b) Show that f is injective if and only if

$$f^{-1}[f(A)] = A$$

for all subsets $A \subseteq X$.

2. Provide a proof or a counterexample to each of the following.

(a) If $f : X \rightarrow Y$ has the property that

$$f(f^{-1}[f(A)]) = f(A)$$

for all subsets $A \subseteq X$, then f is bijective.

(b) If $f : X \rightarrow Y$ has the property that

$$f(f^{-1}[f(f^{-1}[B])]) = B,$$

for all subsets $B \subseteq Y$, then f is bijective.

3. Provide a proof or a counterexample to each of the following.

(a) A function $f : X \rightarrow Y$ is injective if and only if, for all subsets $A, B \subseteq X$,

$$f(A \cap B) = f(A) \cap f(B).$$

(b) A function $f : X \rightarrow Y$ is surjective if and only if, for all subsets $C, D \subseteq Y$,

$$f^{-1}[C \cup D] = f^{-1}[C] \cup f^{-1}[D].$$

4. (a) Prove by induction that, for every positive integer n ,

$$\sum_{k=1}^n k^3 = \frac{n^2(n+1)^2}{4}.$$

(b) Prove that, for every positive integer n ,

$$(1 + 2 + \dots + n)^2 = 1^3 + 2^3 + \dots + n^3.$$

5. (a) Find a formula for

$$\sum_{k=1}^n \frac{1}{2^k},$$

and prove your answer by induction.

- (b) Find a formula for

$$\sum_{k=1}^n \frac{1}{3^k},$$

and prove your answer by induction.

6. Find a formula for the sum of the first n fourth powers, and prove your answer by induction. Also explain how you were able to guess the right formula.
7. Give a proof or a counterexample for each.

- (a) Suppose that $f : X \rightarrow Y$ and $g : X \rightarrow Y$. Then for all subsets $B \subseteq Y$,

$$f(g^{-1}[B]) = g(f^{-1}[B]).$$

- (b) Suppose that $f : X \rightarrow Y$ and $g : X \rightarrow Y$. Then for all subsets $A \subseteq X$,

$$f^{-1}[g(A)] = g^{-1}[f(A)].$$