Homework Assignment 2

Due: Wednesday, September 20

1. (a) Show that the only logical connectives we really need are $\neg$ and $\wedge$ (we can recover the others from these two).

   (b) Write the statement

   $$[(P \Rightarrow Q) \vee R] \Rightarrow [\neg Q \vee (R \wedge P)]$$

   using only the logical connectives $\neg$ and $\wedge$.

2. (a) Define a new logical connective $\uparrow$ by

   $$P \uparrow Q \Leftrightarrow \neg [P \wedge Q] .$$

   Show that the only logical connective we really need is $\uparrow$.

   (b) Define a new logical connective $\downarrow$ by

   $$P \downarrow Q \Leftrightarrow \neg [P \vee Q] .$$

   Show that the only logical connective we really need is $\downarrow$.

   (c) Write the statement $(P \Rightarrow \neg Q) \wedge R$ using only the logical connective $\uparrow$, and then again using only $\downarrow$.

3. Which of the following are true for all sets A, B and C? Provide a proof or a counterexample.

   (a) If $C \cap (A \cap B) = \varnothing$, then $C \cap A = \varnothing$ or $B \cap A = \varnothing$.

   (b) If $C \subseteq A$ and $C \subseteq B$, then $C \subseteq A \cap B$.

   (c) $(A \backslash B) \backslash C = A \backslash (B \backslash C)$.

   (d) If $(A \cap B) \cup C \subseteq A \cap (B \cup C)$, then $C \subseteq A$.

4. (a) Let A, B and X be sets. Show that if $A \cup B \subseteq X$, then

   $$A \backslash B = (X \backslash B) \backslash (X \backslash A).$$

   (b) Suppose that $f : X \to Y$ is a function, and $A, B \subseteq X$. Is it necessarily true that

   $$f(A) \backslash f(B) = f(A \backslash B)?$$

   (c) Suppose that $f : X \to Y$, and $C, D \subseteq Y$. Is it necessarily true that

   $$f^{-1}[C] \backslash f^{-1}[D] = f^{-1}[C \backslash D]?$$

5. Let X and Y be two sets.

   (a) If $x \in X$ and $y \in Y$, show that the ordered pair (x, y) is a subset of $\mathcal{P}(X \cup Y)$.

   (b) Show that the cartesian product $X \times Y$ is an element of the set $\mathcal{P}(\mathcal{P}(\mathcal{P}(X \cup Y)))$.

6. Let A and B be subsets of a set X, let Y be a set, and let $f : X \to Y$ be a map.

   (a) Prove that $f(A \cap B) \subseteq f(A) \cap f(B)$. Must equality hold?

   (b) Show that $f(A \cup B) \subseteq f(A) \cup f(B)$. Must equality hold?

   (c) Is there a relationship between $f(A \vartriangle B)$ and $f(A) \vartriangle f(B)$?

7. Let A be a subset of a set X and let B be a subset of a set Y. Let $f : X \to Y$ be a map.

   (a) Show that $A \subseteq f^{-1}(f(A))$. Must equality hold?

   (b) Show that $f(f^{-1}(B)) \subseteq B$. Show that equality holds if and only if $B \subset f(X)$.

   (c) Find a relationship between $f^{-1}(Y \backslash B)$ and $X \backslash f^{-1}(B)$.