Math 74: Homework Assignment 1

Due: Monday, September 11, 2006

1. **De Morgan's Laws.** Consider the following two statements:

$$\neg(P \vee Q) \iff (\neg P \wedge \neg Q), \tag{1}$$

and

$$\neg(P \wedge Q) \iff (\neg P \vee \neg Q). \tag{2}$$

   (a) Rewrite each of the two statements, using only words (no symbols except P and Q).

   (b) Using truth tables, prove that the two statements are tautologies; that is, they are always true, no matter what P and Q are.

   (c) Let X be a set, and let A and B be subsets of X. Show that

$$X \backslash (A \cup B) = (X \backslash A) \cap (X \backslash B). \tag{3}$$

   (d) Which of the two propositional logic statements above is equation (3) related to, and what is the relationship?

   (e) Discover another set theory fact along the lines of (3), which is the analogue of the other propositional logic statement, and prove it.

2. Which of the following statements are tautologies?

   (a) $[(P \vee Q) \vee R] \iff [P \vee (Q \vee R)]$.

   (b) $[(P \wedge Q) \wedge R] \iff [P \wedge (Q \wedge R)]$.

   (c) $[(P \wedge Q) \vee R] \iff [P \wedge (Q \vee R)]$.

   (d) Explain why we can interpret the statement $P \vee Q \vee R$ as well as $P \wedge Q \wedge R$, but that $P \vee Q \wedge R$ doesn't make sense.

3. Which of the following statements are tautologies? Hint: using De Morgan's Laws and the fact that $(P \implies Q) \iff (\neg P \vee Q)$, you can often write short proofs of tautologies that escape the need for long truth tables.

   (a) $[P \iff Q] \iff [(P \wedge Q) \vee \neg(P \vee Q)]$.

   (b) $[(Q \wedge R) \implies (P \vee T)] \iff [P \vee \neg Q \vee \neg R \vee T]$.

   (c) $[\neg(P \implies \neg(Q \wedge R))] \iff [P \vee Q \vee R]$.

   (d) $[(P \wedge Q) \vee (R \wedge S)] \iff [(P \vee R) \wedge (Q \vee S)]$.

4. Write the negation of each of the following statements. Write your answer clearly, in complete sentences.

   (a) For every integer n, there exists an integer m such that $m = 2n$.

   (b) There exists an integer n such that for all integers m, $m = 2n$.

   (c) For every $y \in B$ there exists an element $x \in A$ such that $f(x) = y$.

   (d) For every $x \in \mathbb{R}$ and for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $y \in \mathbb{R}$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

   (e) There exists a constant $L > 0$ such that $|f(x) - f(y)| < L$ for all $x, y \in \mathbb{R}$.

   (f) For all $n \in \mathbb{N}$, if there exists $m \in \mathbb{N}$ such that $m^2 = n$, then $k^3 \neq n$ for all $k \in \mathbb{N}$.

5. Let A and B be subsets of a set X.

   (a) Prove that $A \subset B$ if and only if A and $X \backslash B$ are disjoint.

   (b) Prove that $A = B$ if and only if $X \backslash A = X \backslash B$.

6. The **symmetric difference** of two sets A and B is the set $A \triangle B$ consisting of those elements x that belong to either A or B, but not both.

   (a) Express $A \triangle B$ in terms of union, intersection, and complement.

   (b) Show that symmetric difference is associative. That is, for any sets A, B, and C, prove that
$$A \triangle (B \triangle C) = (A \triangle B) \triangle C.$$