Math 202B Solutions

Assignment 11

D. Sarason

41. Let A be a finite-dimensional subspace and A' a closed subspace of a Banach space B. Prove that $A + A'$ is closed.

    Proof: Let $\pi : B \to B/A'$ be the quotient map. Then $A + A' = \pi^{-1}(\pi(A))$, where $\pi(A)$ is a closed subspace of B/A' since it's finite-dimensional. Since π is continuous, this implies $A + A'$ is closed also.

42. Let A be the subspace of sequences $(a_n)_1^\infty$ in c_0 such that $a_{2n} = 0$ for all n. Let B be the subspace of sequences $(b_n)_1^\infty$ in c_0 such that $b_{2n} = b_{2n-1}/n$ for all n. Prove that $A + B$ is dense in c_0 but not equal to c_0. (Hence, the vector sum of two closed subspaces of a Banach space need not be closed.)

    Proof: In fact, the sequence $(c_n)_1^\infty$ in c_0 lies in $A+B$ if and only if $nc_{2n} \to 0$. The necessity of the previous condition is obvious. If the condition is satisfied, we define $(b_n)_1^\infty$ by letting $b_{2n} = c_{2n}$ and $b_{2n-1} = nc_{2n}$. Then we define $(a_n)_1^\infty$ by letting $a_{2n-1} = c_{2n-1} - nc_{2n}$ and $a_{2n} = 0$. The sequences $(a_n)_1^\infty$ and $(b_n)_1^\infty$ are in A and B, respectively, and $a_n + b_n = c_n$ for all n.

    From this it is obvious that $A + B \neq c_0$. On the other hand, let $(c_n)_1^\infty$ be in c_0, and fix $\epsilon > 0$. Then for some N, $|c_n| < \epsilon$ for $n \geq N$. Defining $d_n = c_n$ for $n < N$ and $d_n = 0$ for $n \geq N$, $\|c_n - d_n\|_\infty \leq \epsilon$. From the above, it is obvious that $(d_n)_1^\infty \in A + B$, showing that $A + B$ is dense in c_0.

43. Prove that the closed unit ball of an infinite-dimensional Banach space is not compact.

    Proof: Let B be an infinite-dimensional Banach space. Choose any unit vector x_1 in B, and let A_1 be the subspace spanned by x_1. In the quotient space B/A_1, choose a coset of norm $\frac{1}{2}$ and then a representative x_2 of that coset of norm at most 1. Then $\|x_2 - x_1\| \geq \frac{1}{2}$. Let A_2 be the subspace spanned by x_1 and x_2, and note that A_2 is closed by problem 28. In the quotient space B/A_2, choose a coset of norm $\frac{1}{2}$ and then a representative x_3 of that coset of norm at most 1. Then $\|x_3 - x_2\| \geq \frac{1}{2}$ and $\|x_3 - x_1\| \geq \frac{1}{2}$. Continuing in this way, we obtain a sequence $(x_n)_1^\infty$ of vectors in the closed unit ball of B such that $\|x_m - x_n\| \geq \frac{1}{2}$ whenever $m \neq n$.

    The sequence $(x_n)_1^\infty$ then has no convergent subsequences, implying that the closed unit ball in B is not compact.

44. Prove that an infinite-dimensional Banach space cannot be spanned, as a vector space, by a countable subset.

    Proof: We argue by contradiction. Suppose the infinite-dimensional Banach space B is spanned as a vector space by the vectors $x_1, x_2, \ldots$. For each positive integer n let B_n be the subspace spanned by $x_1, \ldots, x_n$. The subspaces $B_1, B_2, \ldots$ are finite-dimensional, hence closed by Problem 38. Since $\bigcup_{n=1}^\infty B_n = B$, it follows by the Baire category theorem that there is an n_0 such that B_{n_0} has a nonempty interior. This is the desired contradiction, because a proper closed subspace of a Banach space is nowhere dense. In particular, if x is a vector in the subspace and y is a vector not in it, then $x + \frac{1}{n}y$ is not in it for all positive integers n, and $x + \frac{1}{n}y \to x$ in norm.

45. Let μ be a positive measure, let $1 \leq p < q < \infty$, and let X be a subspace of $L^p(\mu) \cap L^q(\mu)$ that is closed in both $L^p(\mu)$ and $L^q(\mu)$. Prove that the L^p-norm and the L^q-norm on X are equivalent.

    Proof 1: Let X_p denote X with the L^p-norm and let X_q denote X with the L^q-norm. It is sufficient to show that the identity map of X_p onto X_q is bounded and invertible. By the closed graph theorem, then, it is enough to show that if $(f_n)_1^\infty$ is a sequence in X that converges in the L^p-norm to g and in the L^q-norm to h, then $g = h$ a.e. This follows immediately from the fact that any norm-convergent sequence in $L^p(\mu)$ (or in $L^q(\mu)$) has a subsequence that converges almost everywhere to its limit.

    Proof 2: Let X' be X with the norm $\|f\|' = \|f\|_p + \|f\|_q$. One easily sees that X' is complete. (In particular, a Cauchy sequence in X' is Cauchy in both X_p and X_q, and the limits must be equal by the reasoning of the previous proof.) The identity map $X' \to X_p$ is bounded, injective and surjective, hence invertible by the open mapping theorem. Therefore the norms $\|\cdot\|_p$ and $\|\cdot\|'$ are equivalent on X. For the same reason, the norms $\|\cdot\|_q$ and $\|\cdot\|'$ are equivalent on X.