Math 16B, Section 1 **Spring 2006**
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MIDTERM EXAMINATION 2 – SOLUTIONS

1. Perform the integrations.

(a) $I_1 = \int_0^{\sqrt{\pi}} x\sin(x^2)dx$ (b) $I_2 = \int_0^{\pi} x^2 \sin x \; dx$

Solution. (a) We make the substitution $u = x^2$, $du = 2x\; dx$:

$$\begin{aligned} I_1 &= \frac{1}{2}\int_0^{\pi} \sin u \; du = -\frac{1}{2}\cos u\Big|_0^{\pi} \\ &= -\frac{1}{2}(-1) + \frac{1}{2}(1) = 1. \end{aligned}$$

(b) We integrate by parts twice:

$$\begin{aligned} I_2 &= -\int_0^{\pi} x^2 \; d(\cos x) = -x^2\cos x\big|_0^{\pi} + \int_0^{\pi} \cos x \; d(x^2) \\ &= \pi^2 + 2\int_0^{\pi} x\cos x \; dx = \pi^2 + 2\int_0^{\pi} x \; d(\sin x) \\ &= \pi^2 + 2x\sin x\big|_0^{\pi} - 2\int_0^{\pi} \sin x \; dx \\ &= \pi^2 + 0 + 2\cos x\big|_0^{\pi} = \pi^2 + 2(-1) - 2(1) \\ &= \pi^2 - 4 \end{aligned}$$

2. For the differential equation $y' = -(y+1)^2(t+1)$:

(a) What are the constant solutions, if any?
(b) What is the general solution?
(c) What is the solution satisfying the initial condition $y(0) = 0$?

Solution. (a) $y = -1$

(b) The equation is separable. We can rewrite it as

$$-\frac{y'}{(y+1)^2} = t+1,$$

or, in differential notation, as

$$-\frac{1}{(y+1)^2}dy = (t+1)dt.$$

Integration gives

$$\frac{1}{y+1} = \frac{(t+1)^2}{2} + C.$$

Solving for y, we get

$$y = -1 + \frac{1}{\frac{(t+1)^2}{2} + C},$$

which together with the constant solution $y = -1$ gives the general solution.

(c) If $y(0) = 0$, then

$$0 = -1 + \frac{1}{\frac{1}{2} + C},$$

implying that $C = \dfrac{1}{2}$, and

$$y(t) = -1 + \frac{2}{(t+1)^2 + 1}.$$

3. Roger Rover invests \$300,000 in a real estate trust, which will pay annual interest of 6%, compounded continuously. He arranges for \$1,000 per month to be transferred from his account in the trust to his ex-wife Grouchita's bank account, in payment of alimony.

   (a) Assuming the transfers to Grouchita's account are made continuously, set up a differential equation satisfied by the balance $P(t)$ in Roger's trust account at time t (where t is measured in years, with $t = 0$ corresponding to the inception of the account).

   (b) Find the general solution of the differential equation.

   (c) Find the particular solution describing Roger's account.

   (d) Find an expression for the time it will take for Roger's account to grow to \$400,000.

Solution. (a) $P' = .06P - 12,000$

(b) The equation is linear. In standard form it becomes

$$P' - .06P = -12,000.$$

The integrating factor is $e^{-.06t}$:

$$\frac{d}{dt}(e^{-.06t}P) = -12,000\, e^{-.06t}.$$

Integration gives

$$\begin{aligned} e^{-.06t}P &= \frac{12,000}{.06}e^{-.06t} + C \\ &= 200,000\, e^{-.06t} + C. \end{aligned}$$

The general solution is

$$P(t) = 200,000 + Ce^{.06t}.$$

(c) For Roger's account, $P(0) = 300,000$, implying that $C = 100,000$, and

$$P(t) = 200,000 + 100,000\, e^{.06t}.$$

(d) If $P(t) = 400,000$, then

$$\begin{aligned} 400,000 &= 200,000 + 100,000\, e^{.06t} \\ e^{.06t} &= 2 \\ t &= \frac{\ln 2}{.06}, \text{ the time in years it will take Roger's account to reach \$400,000.} \end{aligned}$$

4. An airliner in the fleet of Krate Airways flies at a constant speed of 10 miles/minute along a straight path at a constant altitude of 5 miles. At noon, the plane is directly above radar station A, which records the angle of elevation $\theta(t)$ of the plane as seen from A, as a function of time. Assuming t is measured in minutes with $t = 0$ corresponding to noon, find $\theta'(1)$ in radians/minute.

Solution. Let $B = B(t)$ be the position of the plane at time t and $C = C(t)$ the point on the ground directly below B. The triangle ABC is a right triangle, with the right angle at the vertex C, and the angle $\theta(t)$ at the vertex A. The side BC has length 5 and the side AC has length $10t$. Hence

$$(*) \qquad \tan\theta(t) = 5/10t = 1/2t.$$

Differentiating, we get

$$\theta'(t)\sec^2(\theta(t)) = \frac{-1}{2t^2}$$

$$(**) \qquad \theta'(t) = \frac{-1}{2t^2\sec^2(\theta(t))}.$$

From $(*)$ we have $\tan\theta(1) = \frac{1}{2}$, so

$$\sec^2\theta(1) = 1 + \tan^2\theta(1) = 1 + \frac{1}{4} = \frac{5}{4}.$$

In combination with $(**)$, this gives

$$\theta'(1) = \frac{-1}{2\left(\frac{5}{4}\right)} = -2/5 \text{ radians/min.}$$