Math 16B, Section 1 **Spring 2006**
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MIDTERM EXAMINATION 1 – SOLUTIONS

1. For which values of a and b does the straight line $y = ax + b$ give the best least-squares approximation to the data points $(0, 0)$, $(1, 2)$, $(2, 3)$?

   Solution. The problem is to minimize the function

   $$E(a, b) = b^2 + (a + b - 2)^2 + (2a + b - 3)^2.$$

   We look for critical points of E. We have

   $$\begin{aligned} \frac{\partial E}{\partial a} &= 2(a + b - 2) + 4(2a + b - 3) = 10a + 6b - 16 \\ \frac{\partial E}{\partial b} &= 2b + 2(a + b - 2) + 2(2a + b - 3) = 6a + 6b - 10 \end{aligned}$$

   The critical points (a, b) are the solutions of the pair of equations

   $$10a + 6b = 16, \qquad 6a + 6b = 10$$

   Subtracting the second equation from the first, we obtain $4a = 6$, or $a = 3/2$. Replacing a by $3/2$ in the second equation, we get $9 + 6b = 10$, or $b = 1/6$.

   Conclusion. There is only one critical point, $(3/2, 1/6)$. The desired straight line is $y = \dfrac{3}{2}x + \dfrac{1}{6}$.

2. For the function $f(x, y) = x^3 + y^3 - 12x - 27y$:

   (a) Determine the critical points.

   (b) Determine which critical points, if any, are saddle points.

   Solution. (a) We have

   $$\frac{\partial f}{\partial x} = 3x^2 - 12, \qquad \frac{\partial f}{\partial y} = 3y^2 - 27.$$

   Setting $\dfrac{\partial f}{\partial x} = 0$, we get the solutions $x = \pm 2$. Setting $\dfrac{\partial f}{\partial y} = 0$, we get the solutions $y = \pm 3$.

   Conclusion. There are four critical points: $(2, 3)$, $(2, -3)$, $(-2, 3)$, $(-2, -3)$.

   (b) We have

   $$\frac{\partial^2 f}{\partial x^2} = 6x, \qquad \frac{\partial^2 f}{\partial y^2} = 6y, \qquad \frac{\partial^2 f}{\partial x \partial y} = 0.$$

   Thus

   $$D_f = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 36xy.$$

   A critical point (a, b) of f is a saddle point if and only if $D_f(a, b) < 0$, which in our case happens if and only if $ab < 0$.

   Conclusion. The critical points $(2, -3)$ and $(-2, 3)$ are saddle points of f. The other two critical points are not saddle points. (For the record, $(2, 3)$ is a relative minimum and $(-2, -3)$ is a relative maximum.)

3. Evaluate the integral $I = \iint_R (x^2 - y)dxdy$, where the region R is defined by the inequalities $0 \le y \le 1$, $x^2 \le y$.

Solution 1. We integrate with respect to x first:

$$
\begin{aligned}
I &= \int_0^1 \left[\int_{-\sqrt{y}}^{\sqrt{y}} (x^2 - y)dx \right] dy = \int_0^1 \left[\left(\frac{x^3}{3} - xy \right) \Bigg|_{-\sqrt{y}}^{\sqrt{y}} \right] dy \\
&= \int_0^1 \left(\frac{y^{3/2}}{3} - y^{3/2} + \frac{y^{3/2}}{3} - y^{3/2} \right) dy = \int_0^1 -\frac{4}{3} y^{3/2} dy \\
&= -\frac{4}{3} \left(\frac{2}{5} y^{5/2} \right) \Bigg|_0^1 = -\frac{8}{15} \text{ (Answer)}
\end{aligned}
$$

Solution 2. We integrate with respect to y first:

$$
\begin{aligned}
I &= \int_{-1}^1 \left[\int_{x^2}^1 (x^2 - y)dy \right] dx = \int_{-1}^1 \left[\left(x^2 y - \frac{y^2}{2} \right) \Bigg|_{x^2}^1 \right] dx \\
&= \int_{-1}^1 \left(x^2 - \frac{1}{2} - x^4 + \frac{x^4}{2} \right) dx \\
&= \int_{-1}^1 \left(x^2 - \frac{1}{2} - \frac{x^4}{2} \right) dx \\
&= \left(\frac{x^3}{3} - \frac{x}{2} - \frac{x^5}{10} \right) \Bigg|_{-1}^1 \\
&= \frac{1}{3} - \frac{1}{2} - \frac{1}{10} + \frac{1}{3} - \frac{1}{2} - \frac{1}{10} = \frac{10 - 15 - 3 + 10 - 15 - 3}{30} \\
&= -\frac{16}{30} = -\frac{8}{15} \text{ (Answer)}
\end{aligned}
$$

4. According to U.S. postal regulations, a rectangular package whose three side lengths measure x inches, y inches, z inches, must satisfy $2x + 2y + z \le 84$. Which values of x, y, z give the dimensions of a package whose diagonal has maximum length? (The length of the diagonal equals $\sqrt{x^2 + y^2 + z^2}$.)

Solution. We use the Lagrange method, rephrasing the problem as: Maximize the function $f(x, y, z) = x^2 + y^2 + z^2$ under the constant $g(x, y) = 84 - 2x - 2y - z = 0$. We introduce the auxiliary variable λ and the auxiliary function

$$
\begin{aligned}
F(x, y, z, \lambda) &= f(x, y, z) + \lambda g(x, y, z) \\
&= x^2 + y^2 + z^2 + \lambda(84 - 2x - 2y - z).
\end{aligned}
$$

We look for critical points of F. We have

$$
\frac{\partial F}{\partial x} = 2x - 2\lambda, \qquad \frac{\partial F}{\partial y} = 2y - 2\lambda, \qquad \frac{\partial F}{\partial z} = 2z - \lambda.
$$

Setting $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$ and solving the resulting equations for λ, we find that $\lambda = x$, $\lambda = y$, $\lambda = 2z$, which tells us that $x = y = 2z$. Substituting $2z$ for x and y in the constraint $2x + 2y + z = 84$ gives us

$$4z + 4z + z = 9z = 84,$$

and accordingly $z = 84/9 = 9\frac{1}{3}$. Thus $x = 2z = 18\frac{2}{3}$, $y = 2z = 18\frac{2}{3}$. We see that F has one critical point, whose x, y, z coordinates are $18\frac{2}{3}, 18\frac{2}{3}, 9\frac{1}{3}$.

Conclusion. The package with longest diagonal measures $18\frac{2}{3}$ inches by $18\frac{2}{3}$ inches by $9\frac{1}{3}$ inches. (The diagonal of this package turns out to be 28 inches long.)