Math 16B – S06 – Supplementary Notes 1
The Derivative and Local Linear Approximation

For a function $g(x)$ of one variable, the value $g'(a)$ of the derivative of g at the point $x = a$ is the slope of the tangent line to the graph of g at the point $(a, g(a))$. Near that point, the tangent line is a "good" approximation to the graph, in the following sense. The equation of the tangent line is

$$y = g(a) + g'(a)(x - a).$$

The difference

$$R(x) = g(x) - g(a) - g'(a)(x - a) \tag{1}$$

is the error you get at x when you approximate g by the function whose graph is the tangent line (see Figure 1.1). The approximation is good near a in the sense that $R(x)$ is small for x near a compared to $x - a$, vanishingly small in the limit:

$$\lim_{x \to \infty} \frac{R(x)}{x - a} = 0. \tag{2}$$

To see this, simply divide (1), the equality that defines $R(x)$, by $x - a$, to get

$$\frac{R(x)}{x - a} = \frac{g(x) - g(x)}{x - a} - g'(a).$$

As x approaches a, the difference quotient on the right side of the equality approaches $g'(a)$, so the ratio on the left side tends to 0.

Among all straight lines through the point $(a, g(a))$, the tangent line gives the best linear approximation near a to the graph of g. For the line with slope $c \neq g'(a)$, the error in the approximation at x, divided by $x - a$, has the limit $g'(a) - c$, not 0, as x approaches a.

The relation (2), while it says you can expect the error in a certain approximation to be small, does not tell you how small. It does not tell you, for example, how close x should be to a to guarantee that the approximation is accurate to, say, three decimal places. Later in the course, when we study Taylor polynomials, we will learn of a method for obtaining useful error estimates. (For linear approximations of g the estimate involves the second derivative of g.) At this point one can obtain a good feel for the situation by looking at a few examples. Here is one example. We take $g(x) = \sqrt{x}$ and $a = 4$. Then $g(4) = 2$, $g'(x) = 1/2\sqrt{x}$, and $g'(4) = 1/4$. The approximating linear function is the function $2 + \frac{1}{4}(x - 4)$, which we expect to give a good approximation to $\sqrt{x}$ for x near 4. The following table compares $\sqrt{4}$, given to four decimal places, to the approximate value $2 + \frac{1}{4}(x - 4)$, for various values of x near 4.

x	$2+\frac{1}{4}(x-4)$	$\sqrt{x}$	$R(x)$
3	1.75	1.7321	$-.0179$
3.5	1.875	1.8708	$-.0042$
3.6	1.9	1.8974	$-.0026$
3.7	1.925	1.9235	$-.0015$
3.8	1.95	1.9494	$-.0006$
3.9	1.975	1.9748	$-.0002$
4.1	2.025	2.0248	$-.0002$
4.2	2.05	2.0494	$-.0006$
4.3	2.075	2.0736	$-.0019$
4.4	2.1	2.0976	$-.0024$
4.5	2.125	2.1213	$-.0037$
5	2.25	2.2361	$-.0139$

One sees that for $|x-4| \leq .5$ the approximation is good to two decimals, for $|x-4| \leq .1$ it is good to three.

Question. Why in this example does the approximation always produce an overestimate?

More Variables

The considerations above, with partial derivatives in place of derivatives, apply to functions of several variables. Consider a function $f(x, y)$ of two variables. Near a point (a, b) in the plane consider the two one-variable functions $f(x, b)$ and $g(a, y)$. We then have the approximation

$$f(x,b) \approx f(a,b) + \left(\frac{\partial f}{\partial x}(a,b)\right)(x-a) \tag{3}$$

$$f(a,y) \approx f(a,b) + \left(\frac{\partial f}{\partial y}(a,b)\right)(y-b), \tag{4}$$

the first holding for x near a, the second for y near b.

The relations (3) and (4) are special cases of a more general linear approximation property, namely

$$f(x,y) \approx f(a,b) + \left(\frac{\partial f}{\partial x}(a,b)\right)(x-a) + \left(\frac{\partial f}{\partial y}(a,b)\right)(y-b), \tag{5}$$

where the error in the approximation, the quantity

$$R(x,y) = f(x,y) - f(a,b) - \left(\frac{\partial f}{\partial x}(a,b)\right)(x-a) - \left(\frac{\partial f}{\partial y}(a,b)\right)(y-b),$$

is small compared to the distance of (x, y) from (a, b):

$$\lim_{(x,y)\to(a,b)} \frac{R(x,y)}{\sqrt{(x-a)^2+(y-b)^2}} = 0. \tag{6}$$

This is more subtle than its one-variable forebear in that its validity requires more than the mere existence of the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at (a, b). The kinds of functions for which it fails, however, do not arise in Math 16B.

Tangent Lines to Level Curves

Consider a level curve $f(x,y)=c$ of the function f and a point (a,b) on it. The equation for the tangent line to the level curve at (a,b) can (usually) be written in terms of the partial derivatives of f at (a,b). To simplify the notation we shall write f_x for $\frac{\partial f}{\partial x}$ and f_y for $\frac{\partial f}{\partial y}$.

Theorem. *If (a,b) is not a critical point of f, then the tangent line at (a,b) to the level curve of f through (a,b) has the equation*

$$f_x(a,b)(x-a)+f_y(a,b)(y-b)=0. \tag{7}$$

To get an idea of why this is true, consider a point (x_1,y_1) on the level curve close to but distinct from (a,b). We write down the approximate equality (5) for $(x,y)=(x_1,y_1)$. Since we have $f(x_1,y_1)=f(a,b)$, we get

$$f_x(a,b)(x_1-a)+f_y(a,b)(y_1-b)\approx 0,$$

which seems to be saying that the point (x_1,y_1) is nearly on the line (7). To make this more precise we employ the limit relation (6), which in our case reads

$$\lim_{(x_1,y_1)\to(a,b)}\frac{f_x(a,b)(x_1-a)+f_y(a,b)(y-b)}{\sqrt{(x_1-a)^2+(y_1-b)^2}}=0, \tag{8}$$

it being understood that the limit is taken as (x_1,y_1) approaches (a,b) along the level curve. To interpret (8) we introduce the point $(\tilde{x}_1,\tilde{y}_1)$ defined by

$$\tilde{x}_1=a+\frac{x_1-a}{\sqrt{(x_1-a)^2+(y_1-b)^2}},\quad \tilde{y}_1=b+\frac{y_1-b}{\sqrt{(x_1-a)^2+(y_1-b)^2}}.$$

The point $(\tilde{x}_1,\tilde{y}_1)$ lies on the secant line to the level curve determined by the points (a,b) and (x_1,y_1), and it has distance 1 from (a,b) (see Figure 1.2). In terms of $(\tilde{x}_1,\tilde{y}_1)$ we can rewrite (8) as

$$\lim_{(x_1,y_1)\to(a,b)} f_x(a,b)(\tilde{x}_1-a)+f_y(a,b)(\tilde{y}_1-b)=0. \tag{9}$$

As (x_1,y_1) approaches (a,b) along the level curve, the secant line determined by (a,b) and (x_1,y_1) will rotate and approach the tangent line at (a,b). The point $(\tilde{x}_1,\tilde{y}_1)$ will converge to a point $(\tilde{x},\tilde{y})$ on the tangent line, at distance 1 from (a,b), hence different from (a,b). The limit relation (9) tells us that

$$f_x(a,b)(\tilde{x}-a)+f_y(a,b)(\tilde{y}-b)=0,$$

in other words, $(\tilde{x},\tilde{y})$ lies on the line (7). Since a line in the plane is uniquely determined by any two of its points, we conclude that the tangent line is in fact the line (7).

The case where (a,b) is a critical point of f, ignored by the theorem above, is more complicated and will not be discussed here. It will not arise in later developments.