Math 16B, Section 1 **Spring 2006**
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REVIEW EXERCISES 3

1. In each part, find and classify (by means of the second-derivative test) the critical points of the function f.
(a) $f(x, y) = 8x^3 + y^3 - xy$ (b) $f(x, y) = 5x^2 + y^2 - x - y - 2xy$

2. Which pair of values a, b minimizes the function

$$E(a, b) = \int_0^1 (ax + b - x^2)^2 dx?$$

3. Among all rectangular boxes of side lengths x, y, z inches and diagonal measuring 28 inches, which values of x, y, z maximize $2x + 2y + z$?

4. (Final Exam, F04) The Mexican firm Novedades Terminador SSA manufactures Arnold masks. The number of masks it can produce in a week with the utilization of x units of labor and y units of capital is given by the production function $f(x, y) = 500x^{3/5}y^{2/5}$. The expense for each unit of labor is 100 pesos per week and for each unit of capital it is 200 pesos per week. Total expenses are limited to 1000 pesos per week. How many units of labor and capital should the firm utilize so as to maximize production?

5. Evaluate the integrals.
(a) $\displaystyle\int_0^{\pi/2} \sin^3 x \cos^3 x \; dx$ (b) $\displaystyle\int_0^{\infty} x^3 e^{-x^2} \; dx$ (c) $\displaystyle\int_0^1 (x + 1)\ln(x + 1)dx$
(d) $\displaystyle\iint_R \ln(x + y + 1)dxdy$, where R is the triangle with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$.

6. For each part, find the general solution of the differential equation, as well as the particular solution or solutions satisfying the given initial condition or conditions.
   (a) $y' = .065y - 26,000$; Initial condition: $y(0) = 300,000$.
   (b) $2y' = (e - y)^3 e^t$; Initial conditions: $y(0) = e - 1$ and $y(0) = e$.
   (c) $y' = 2y^2 \sin t \cos t$; Initial conditions: $y\left(\frac{\pi}{2}\right) = 1$ and $y\left(\frac{\pi}{2}\right) = 0$.

7. Horace and Hortense Snootee set up a college fund for their son Rudy on the day of his birth. They start the fund with a \$20,000 deposit and will add \$7,000 per year. The fund is expected to grow at the rate of 7% per year, compounded continuously.
   (a) Assuming the Snootee's yearly additions to the fund are made continuously, set up a differential equation satisfied by the amount $P(t)$ in the fund t years after its inception.
   (b) Find the general solution of the differential equation.
   (c) Find the solution of the differential equation satisfying the initial condition $P(0) = 20,000$.
   (d) Determine the balance in the fund on Rudy's 18th birthday.

8. (Final Exam, F04) For the function $f(x) = \sqrt{x}$:
   (a) Find the second Taylor polynomial $p_2(x)$ at $x = 100$.

(b) Evaluate $p_2(104)$, and use the remainder estimate to estimate how closely $p_2(104)$ approximates $\sqrt{104}$.

9. (Final Exam, F05) (a) Find the third Taylor polynomial $p_3(x)$ at $x = 0$ for the function $f(x) = \ln(1 + x)$.

   (b) Use the result from (a) to estimate $\ln(1.1)$.

   (c) Use the remainder estimate to get an upper bound for the error made in (b).

10. (a) For the function $f(x) = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6}$, find the third Taylor polynomial $p_3(x)$ at $x = 1$.

    (b) For the same function, find the third Taylor polynomial at $x = 0$.

    (c) In what way do the polynomials found in parts (a) and (b) differ?

11. (a) Find the Taylor series at $x = 0$ for the function $f(x) = 1/(1 - x)^3$.

    (b) Use the result from (a) to evaluate $\displaystyle\sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2^n}$.

12. (a) Write the repeating decimal $.909090\ldots$ as a fraction.

    (b) Find the third Taylor polynomial $p_3(x)$ at $x = 1$ for the function $f(x) = \dfrac{1}{x}$.

    (c) Use $p_3(x)$ from part (b) to estimate $\dfrac{1}{1.1}$. Bound the error using the remainder estimate.

    (d) Explain the connection between parts (a) and (c).

13. Let the random variable X denote the outcome of rolling a fair pair of dice. The possible values of X are thus $2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$.

    (a) Determine the probability of each possible value of X.

    (b) Compute the probabilities $Pr(X < 7)$, $Pr(6 \le X \le 8)$, $Pr(X \text{ is odd})$.

    (c) Suppose someone gives you 2-to-1 odds that you will not roll 6, 7 or 8. If the bet is for \$1, what is the expected value of your winnings or losses, as the case may be.

14. (Final Exam, F05) For a continuous random variable X with probability density function $f(x) = \dfrac{3}{2}\sqrt{x}$, $0 \le x \le 1$, compute the expected value $E(X)$ and the variance $\mathrm{Var}(X)$.

15. An opaque bag contains 50 marbles of the same size, 30 green and 20 red. You reach into the bag and, without looking inside, extract a marble. The experiment is considered a success if the extracted marble is green, a failure if it is red. In the case of success, you return the marble to the bag, shake the contents, and repeat the experiment, and so on until the first failure. Let the random variable X be the number of successes before the first failure.

    (a) Determine the probabilities $p_n = Pr(X = n)$, $n = 0, 1, \ldots$.

    (b) Using the result from part (a), write down an infinite series that expresses the expected value $E(X)$.

    (c) Find the Taylor series at $x = 0$ for the function $f(x) = x/(1 - x)^2$, and use it to sum the series found in part (b).