Math 16B, Section 1 **Spring 2006**
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FINAL EXAMINATION SOLUTIONS

1. Evaluate the integrals:

   (a) $I_1 = \iint_R e^{x+y} dx\ dy$, where R is the triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$.

   (b) $I_2 = \int_0^\infty xe^{-x^2} dx$ (c) $I_3 = \int_0^{\pi^2} \sin\sqrt{x} dx$

   Solution. (a) Since $e^{x+y} = e^x e^y$, we have

$$
\begin{aligned}
I_1 &= \int_0^1 e^x \left[\int_0^x e^y dy\right] dx = \int_0^1 e^x (e^y)\Big|_{y=0}^{y=x} dx \\
&= \int_0^1 e^x(e^x - 1)dx = \int_0^1 (e^{2x} - e^x)dx \\
&= \left(\frac{1}{2}e^{2x} - e^x\right)\Big|_0^1 = \frac{1}{2}e^2 - e - \frac{1}{2} + 1 \\
&= \boxed{\frac{1}{2}e^2 - e + \frac{1}{2}}
\end{aligned}
$$

   (b) Using the substitution $u = x^2$, $du = 2x\ dx$, we get, for $b > 0$,

$$
\begin{aligned}
\int_0^b xe^{-x^2} dx &= \frac{1}{2}\int_0^{b^2} e^{-u} du = -\frac{1}{2}e^{-u}\Big|_0^{b^2} \\
&= \frac{1}{2}(1 - e^{-b^2}) \underset{(b\to\infty)}{\longrightarrow} \frac{1}{2}.
\end{aligned}
$$

   Thus

$$
I_2 = \lim_{b\to\infty} \int_0^b xe^{-x^2} dx = \boxed{\frac{1}{2}}
$$

   (c) We make the substitution $x = u^2$, $dx = 2du$, and then integrate by parts:

$$
\begin{aligned}
I_3 &= 2\int_0^\pi u \sin u\ du = -2\int_0^\pi u\ d(\cos u) \\
&= (-2u\cos u)\big|_0^\pi + 2\int_0^\pi \cos u\ du \\
&= 2\pi + (2\sin u)\big|_0^\pi = \boxed{2\pi}.
\end{aligned}
$$

2. Let

$$
E(a,b) = \iint_R [(x-a)^2 + (y-b)^2] dx\ dy,
$$

   where R is the square with vertices $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$. For which (a,b) is $E(a,b)$ a minimum?

Solution. We have

$$
\begin{aligned}
E(a,b) &= \int_0^1 \left[\int_0^1 ((x-a)^2+(y-b)^2)dy\right] dx \\
&= \int_0^1 \left(y(x-a)^2+\frac{(y-b)^3}{3}\right)\Bigg|_{y=0}^{y=1} dx \\
&= \int_0^1 \left((x-a)^2+\frac{(1-b)^3}{3}+\frac{b^3}{3}\right) dx \\
&= \frac{(x-a)^3}{3}\Bigg|_0^1+\frac{(1-b)^3}{3}+\frac{b^3}{3} \\
&= \frac{(1-a)^3}{3}+\frac{a^3}{3}+\frac{(1-b)^3}{3}+\frac{b^3}{3}.
\end{aligned}
$$

Hence

$$\frac{\partial E}{\partial a} = -(1-a)^2+a^2 = 2a-1, \qquad \frac{\partial E}{\partial b} = -(1-b)^2+b^2 = 2b-1.$$

We see that $\left(\frac{1}{2},\frac{1}{2}\right)$ is the unique critical point of $E(a,b)$, hence is the minimum of $E(a,b)$.

3. The Pauvre Suceur Gambling Accessories Manufacturing Company has a contract to produce 960,000 decks of cards. For the plant where the cards are made, the production function $f(x,y) = 12,000x^{2/3}y^{1/3}$ gives the number of decks that can be produced with the utilization of x units of labor and y units of capital. Each unit of labor costs \$1,000 and each unit of capital costs \$4,000.

   (a) Write down the function $g(x,y)$ giving the cost to the company when it utilizes x units of labor and y units of capital.

   (b) Determine the values of x and y that minimize the cost of producing 960,000 decks of cards. Use Lagrange's method and take care not to confuse the objective and constraint functions. (You will lose points if you do confuse them.)

   (c) Compare labor costs with capital costs for the minimizing values of x and y.

Solution. (a) $g(x,y) = 1000x + 4000y$

(b) The problem is to minimize $g(x,y)$ under the constraint $f(x,y) = 960,000$. Using Lagrange's method, we introduce the function

$$F(x,y,\lambda) = 1000x+4000y+\lambda(960,000-12,000x^{2/3}y^{1/3}),$$

and we look for the x and y coordinates of its critical points in the region $x>0$, $y>0$. We have

$$
\begin{aligned}
\frac{\partial F}{\partial x} &= 1000-8000\lambda x^{-1/3}y^{1/3} \\
\frac{\partial F}{\partial y} &= 4000-4000\lambda x^{2/3}y^{-2/3}.
\end{aligned}
$$

Setting $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ equal to 0 and solving for λ, we obtain

$$\lambda = \frac{1}{8}x^{1/3}y^{-1/3}, \qquad \lambda = x^{-2/3}y^{2/3}.$$

Eliminating λ, we find that $x = 8y$ at a critical point. Substituting $8y$ for x in our constraint, we obtain

$$960,000 = 12,000(8y)^{2/3}y^{1/3} = 48,000y,$$

giving $y = 20$, $x = 8y = 160$. We can conclude that minimum cost is achieved when 160 units of labor and 20 units of capital are utilized.

(c) We have

$$\frac{\text{labor costs}}{\text{capital costs}} = \frac{(160)(1000)}{(20)(4000)} = 2.$$

4. (a) Find the general solution of the differential equation

$$2yy' = -(y^2-1)^2.$$

(b) Find the solution satisfying the initial condition $y(1) = -2$.

(c) Find the solution satisfying the initial condition $y(1) = -1$.

Solution. (a) We see by inspection that there are two constant solutions, $y = 1$ and $y = -1$. The equation is separable. We can rewrite it as

$$-\frac{2yy'}{(y^2-1)^2} = 1,$$

or, in differential notation, as

$$-\frac{2y}{(y^2-1)^2}dy = dt.$$

Integrating, we get

$$-\int \frac{2y}{(y^2-1)^2}dy = \int dt = t + C.$$

To perform the integration on the left side, we make the substitution $u = y^2 - 1$, $du = 2ydy$:

$$-\int \frac{2y}{(y^2-1)^2}dy = -\int \frac{1}{u^2}du = \frac{1}{u} = \frac{1}{y^2-1}.$$

(We can ignore the constant of integration here, because it can be incorporated into the constant C on the right side of the equality.) We get

$$\begin{aligned} \frac{1}{y^2-1} &= t + C, \\ y &= \pm\sqrt{1 + \frac{1}{t+C}}, \end{aligned}$$

which together with the constant solutions $y = 1$, $y = -1$, gives the general solution of the equation.

(b) If $y(1) = -2$ then the minus sign in the preceding expression will be in effect. We obtain

$$\begin{aligned} -2 &= -\sqrt{1 + \frac{1}{1+C}} \\ 4 &= 1 + \frac{1}{1+C} \\ C &= -\frac{2}{3}. \end{aligned}$$

The solution satisfying $y(1) = -2$ is

$$y = -\sqrt{1 + \frac{1}{t - \frac{2}{3}}}.$$

(c) If $y(1) = -1$, then y is the constant function $y = -1$.

5. Bianca Confucion takes out a \$500,000 mortgage to buy a hovel near the Berkeley campus. The yearly interest rate is 5%, compounded continuously, and yearly payments are \$35,000, applied continuously.

   (a) Set up a differential equation satisfied by the unpaid amount $P(t)$ of the mortgage at time t (with t measured in years).

   (b) Find the general solution of the differential equation.

   (c) Find the solution satisfying the initial condition $P(0) = 500,000$.

   (d) Determine how long it will take Bianca to repay the loan in full. (You will need to use the logarithm table on the cover sheet.)

Solution. (a) $P' = .05P - 35,000$.

(b) The equation is linear. In standard form it becomes

$$P' - .05P = -35,000.$$

Multiplying by the integrating factor $e^{-.05t}$ and integrating, we obtain

$$\begin{aligned} e^{-.05t}P &= -\int 35,000e^{-.05t}dt \\ &= \frac{-35,000e^{-.05t}}{-.05} + C \\ &= 700,000e^{-.05t} + C. \end{aligned}$$

Thus our general solution is

$$P = 700,000 + Ce^{.05t}.$$

(c) If $P(0) = 500,000$ then $C = -200,000$, and our solution becomes

$$P = 700,000 - 200,000e^{.05t}.$$

(d) If t_0 is the time at which the loan becomes fully repaid, then $P(t_0) = 0$, so

$$\begin{aligned} e^{.05t_0} &= \frac{7}{2}, \\ t_0 &= \frac{\ln\frac{7}{2}}{.05} = \frac{\ln 7 - \ln 2}{.05} \\ &= \frac{1.9459 - .6931}{.05} \text{ (from the table)} \\ &= (20)(1.2528) = \boxed{25.056 \text{ years}} \end{aligned}$$

CORRECTION. On the cover sheet of the exam, an incorrect value (.6391) is given for ln 2. In the solution of part (d) of question 5 the correct value (.6931) is used.

6. (a) Find the third Taylor polynomial $p_3(x)$ at $x = 1$ for the function $f(x) = \sqrt{x}$.

(b) Use the result from (a) to estimate $\sqrt{1.2}$. Express your answer in decimal form.

(c) Use the remainder estimate to get a bound on the error in the approximation obtained in part (b). Again, express your answer in decimal form.

Solution. (a) We have

$$\begin{aligned} f(x) &= x^{1/2}, \qquad f'(x) = \frac{1}{2}x^{-1/2}, \qquad f''(x) = -\frac{1}{4}x^{-3/2}, \qquad f^{(3)}(x) = \frac{3}{8}x^{-5/2} \\ f(1) &= 1, \qquad f'(1) = \frac{1}{2}, \qquad f''(1) = -\frac{1}{4}, \qquad f^{(3)}(1) = \frac{3}{8}. \end{aligned}$$

We obtain

$$p_3(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3.$$

(b)

$$\begin{aligned} \sqrt{1.2} &\approx p_3(1.2) = 1 + \frac{1}{2}(.2) - \frac{1}{8}(.2)^2 + \frac{1}{16}(.2)^3 \\ &= 1 + .1 - \frac{1}{8}(.04) + \frac{1}{16}(.008) \\ &= 1 + .1 - .005 + .0005 = \boxed{1.0955}. \end{aligned}$$

(c) We have $f^{(4)}(x) = -\frac{15}{16}x^{-7/2}$, which in absolute value is at most $\frac{15}{16}$ for $x \geq 1$. By the remainder estimate, the error in the approximation from (b) is bounded by

$$\begin{aligned} \frac{\frac{15}{16}(.2)^4}{4!} &= \frac{15(.0016)}{(16)(4)(3)(2)} = \frac{5(.0001)}{8} \\ &= (.625)(.0001) = \boxed{.0000625}. \end{aligned}$$

7. For a continuous random variable X with probability density function $f(x) = \sin 2x$, $0 \leq x \leq \dfrac{\pi}{2}$, compute the expected value $E(X)$ and the variance $\mathrm{Var}(X)$.

Solution. We use integration by parts to perform the required integrations:

$$\begin{aligned} E(X) &= \int_0^{\pi/2} x \sin 2x \; dx = -\frac{1}{2}\int_0^{\pi/2} x \; d(\cos 2x) \\ &= -\frac{1}{2}x\cos 2x\Big|_0^{\pi/2} + \frac{1}{2}\int_0^{\pi/2} \cos 2x \; dx \\ &= \frac{\pi}{4} + \frac{1}{4}\sin 2x\Big|_0^{\pi/2} = \boxed{\frac{\pi}{4}} \end{aligned}$$

$$\begin{aligned}
E(X^2) &= \int_0^{\pi/2} x^2 \sin 2x \ dx = -\frac{1}{2}\int_0^{\pi/2} x^2 \ d(\cos 2x) \\
&= -\frac{1}{2}x^2 \cos 2x\Big|_0^{\pi/2} + \frac{1}{2}\int_0^{\pi/2} \cos 2x \ d(x^2) \\
&= \frac{\pi^2}{8} + \int_0^{\pi/2} x \cos 2x \ dx \\
&= \frac{\pi^2}{8} + \frac{1}{2}\int_0^{\pi/2} x \ d(\sin 2x) \\
&= \frac{\pi^2}{8} + \frac{1}{2}x \sin 2x\Big|_0^{\pi/2} - \frac{1}{2}\int_0^{\pi/2} \sin 2x \ dx \\
&= \frac{\pi^2}{8} + \frac{1}{4}\cos 2x\Big|_0^{\pi/2} = \frac{\pi^2}{8} - \frac{1}{4} - \frac{1}{4} \\
&= \frac{\pi^2}{8} - \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(x) &= E(X^2) - E(X)^2 = \frac{\pi^2}{8} - \frac{1}{2} - \frac{\pi^2}{16} \\
&= \boxed{\frac{\pi^2}{16} - \frac{1}{2}}
\end{aligned}$$

8. Suppose the possible values of the discrete random variable X range over the nonnegative integers, and the associated probabilities are given by $p_n = Pr(X = n) = 6^n/7^{n+1}$ $(n = 0, 1, 2, \dots)$. Compute $Pr(X \text{ is even})$.

Solution. We have

$$Pr(X \text{ is even}) = \sum_{n=0}^{\infty} p_{2n} = \sum_{n=0}^{\infty} \frac{6^{2n}}{7^{2n+1}}.$$

The infinite series on the right side is a geometric series with initial term $1/7$ and ratio $6^2/7^2$. Thus

$$Pr(X \text{ is even}) = \frac{1/7}{1 - \frac{36}{49}} = \boxed{\frac{7}{13}}.$$

9. (a) Derive the formula

$$\int_a^b x^2 e^{-x^2/2} dx = \int_a^b e^{-x^2/2} dx + ae^{-a^2/2} - be^{-b^2/2}.$$

(b) Let X be a standard normal random variable, i.e., a continuous random variable whose density function is the function $\psi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, $-\infty < x < \infty$. Use the result from (a) to show that $\mathrm{Var}(X) = 1$.

Solution. (a) We integrate by parts:

$$\begin{aligned}\int_a^b x^2 e^{-x^2/2} dx &= -\int_a^b x\ d(e^{-x^2/2}) \\ &= -x\ e^{-x^2/2}\Big|_a^b + \int_a^b e^{-x^2/2} dx \\ &= -be^{-b^2/2} + ae^{-a^2/2} + \int_a^b e^{-x^2/2} dx.\end{aligned}$$

(b) Since $E(X) = 0$ ($x\psi(x)$ being an odd function), $\mathrm{Var}(X) = E(X^2)$. For $b > 0$ we obtain, by (a),

$$\begin{aligned}\int_{-b}^b x^2\ \psi(x) dx &= \frac{1}{\sqrt{2\pi}} \int_{-b}^b x^2 e^{-x^2/2} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-b}^b e^{-x^2/2} dx - \frac{2be^{-b^2/2}}{\sqrt{2\pi}} \\ &= \int_{-b}^b \psi(x) dx - \frac{2be^{-b^2/2}}{\sqrt{2\pi}}.\end{aligned}$$

As $b \to \infty$, the first summand on the right side tends to 1 (since ψ is a probability density function) and the second summand tends to 0, giving the desired conclusion:

$$1 = \int_{-\infty}^{\infty} x^2 \psi(x) dx = \mathrm{Var}(X).$$