Math 16B – F05 – Supplementary Notes 6
Error Estimate for Approximation by Taylor Polynomials

Recall that the n^{th} Taylor polynomial for the function $f(x)$ at the point $x = a$ is the polynomial

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

It is the unique polynomial of degree at most n that agrees with f at the point a and whose first n derivatives agree with those of f at the point a. With the summation notation it can be rewritten as

$$\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k,$$

with the conventions $f^{(0)} = f$ and $0! = 1$.

The n^{th} Taylor polynomial for f at a not only agrees with f at a, also its rate of change at a agrees with that of f, and the same is true for the rates of change of the first $n-1$ derivatives. It is thus reasonable to expect that the Taylor polynomial will approximate f closely for x near a. To quantify this expectation one needs an estimate for the error in the approximation.

The difference between f and its n^{th} Taylor polynomial at a is given by

$$R_n(x,a) = f(x) - f(a) - f'(a)(x-a) - \frac{f''(a)}{2}(x-a)^2 - \frac{f^{(3)}(a)}{3!}(x-a)^3 - \cdots - \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

This is the error in the approximation. Often it is referred to as the remainder in the n^{th} Taylor approximation. How can we obtain a useful estimate of the size of $R_n(x,a)$?

The simplest case is the case $n = 0$, 0^{th}-order Taylor approximation. In this case we are approximating f by the constant function $f(a)$, ridiculous perhaps, but nevertheless indicative of the general case. We have

$$R_0(x,a) = f(x) - f(a) = \int_a^x f'(t)dt.$$

If M as an upper bound of $|f'(t)|$ for t between a and x, then the preceding integral has absolute value at most $M|x-a|$, i.e., $|R_0(x,a)| \le M|x-a|$. As we shall see, a similar estimate holds in the general case.

Let's look at the next simplest case, the case $n = 1$, 1^{st}-order (i.e., linear) approximation. We have

$$R_1(x,a) = f(x) - f(a) - f'(a)(x-a).$$

We now do something clever: instead of keeping the center of approximation a fixed, we let it be variable. We introduce the function

$$R_1(x,t) = f(x) - f(t) - f'(t)(x-t)$$

of the two variables x and t. We differentiate $R(x,t)$ with respect to t:

$$\begin{aligned}\frac{\partial R_1(x,t)}{\partial t} &= -f'(t) - \frac{\partial}{\partial t}(f'(t)(x-t)) \\ &= -f'(t) - f''(t)(x-t) + f'(t) = -f''(t)(x-t).\end{aligned}$$

Now we integrate with respect to t from a to x to get

$$R_1(x,x) - R_1(x,a) = -\int_a^x f''(t)(x-t)dt.$$

But $R(x, x) = 0$, so

$$R_1(x, a) = \int_a^x f''(t)(x - t)dt.$$

If now M is an upper bound of $|f''(t)|$ between a and x, then the absolute value of the preceding integral is at most

$$M \left| \int_a^x |x - t|dt \right| = \frac{M|x - a|^2}{2},$$

giving us the error estimate

$$|R_1(x, a)| \leq \frac{M|x - a|^2}{2}.$$

The preceding method works in the general case to give the following result.

Theorem. $R_n(x, a) = \dfrac{1}{n!} \displaystyle\int_a^x f^{(n+1)}(t)(x - t)^n dt.$

The preceding expression is called the integral form of the remainder in Taylor's formula. Before deriving it, let's note the following consequence.

Corollary. *If $|f^{(n+1)}(t)|$ is bounded by M for t between a and x, then*

$$R_n(x, a) \leq \frac{M|x - a|^{n+1}}{(n + 1)!}.$$

In fact, if M is an upper bound for $|f^{(n+1)}(t)|$ for t in the interval with endpoints a and x, then the integral in the expression for $R_n(x, t)$ is in absolute value no larger than

$$M \left| \int_a^x |x - t|^n dt \right| = \frac{M|x - a|^{n+1}}{n + 1}.$$

To derive the integral formula for the remainder, we look at the function

$$(1) \quad R_n(x, t) = f(x) - f(t) - f'(t)(x - t) - \frac{f''(t)}{2}(x - t)^2 - \frac{f^{(3)}(t)}{3!}(x - t)^3 - \cdots - \frac{f^{(n)}(t)}{n!}(x - t)^n$$

of the two variables x and t. We take the partial derivative with respect to t. There are $n + 2$ summands on the right side of (1). The first of those is $f(x)$, which is independent of t, so its derivative with respect to t is 0. The second is $-f(t)$, whose derivative with respect to t is $-f'(t)$. The third is $-f'(t)(x - t)$, whose derivative with respect to t is $-f''(t)(x - t) + f'(t)$. Note that the sum of the derivatives with respect to t of the first three summands is just $-f''(t)(x - t)$ (as we found when treating the case $n = 1$). The fourth summand is $-\frac{f''(t)}{2}(x - t)^2$, whose derivative with respect to t is $-\frac{f^{(3)}(t)}{2}(x - t)^2 + f''(t)(x - t)$. The sum of the derivatives with respect to t of the first four summands is thus $-\frac{f^{(3)}(t)}{2}(x - t)^2$. This pattern repeats as we go along. The $(k + 2)^{nd}$ summand is $-\frac{f^{(k)}(t)}{k!}(x - t)^k$, whose derivative with respect to t is

$$-\frac{f^{(k+1)}(t)}{k!}(x - t)^k + \frac{f^{(k)}(t)}{(k - 1)!}(x - t)^{k-1}.$$

In the derivative of the $(k+1)^{st}$ summand, the second of these two terms occurs with a minus sign, producing a cancellation. And the first of the two terms appears in the derivative of the $(k+3)^{rd}$ summand, but with the opposite sign, producing another cancellation — except for $k=n$, because the $(n+2)^{nd}$ summand is the last. Adding everything together, we get

$$\frac{\partial R_n(x,t)}{\partial t} = -\frac{f^{(n+1)}(t)(x-t)^n}{n!}.$$

Integration now gives

$$R_n(x,x) - R_n(x,a) = -\frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n dt,$$

which yields the expression in the theorem since $R(x,x)=0$.

The following examples illustrate use of the corollary in estimating errors.

Example 1. $f(x) = \cos x$, $a=0$.

The first six Taylor polynomials of $\cos x$ at the origin are

$$\begin{aligned} p_0(x) &= p_1(x) = 1 \\ p_2(x) &= p_3(x) = 1 - \frac{x^2}{2} \\ p_4(x) - p_5(x) &= 1 - \frac{x^2}{2} + \frac{x^4}{24}. \end{aligned}$$

All derivatives of $\cos x$ are bounded in absolute value by 1, so in the corollary we can take $M=1$ to get, for example

$$|R_5(x,0)| = \left|\cos x - \left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right)\right| \leq \frac{|x|^6}{6!} = \frac{|x|^6}{720}.$$

When x is small the error is quite small. For instance,

$$\left|R_5\left(\frac{1}{2},0\right)\right| \leq \frac{1}{46080} \leq .000022,$$

from which we get

$$\cos\frac{1}{2} \approx 1 - \frac{(1/2)^2}{2} + \frac{(1/2)^4}{24} \approx .87760,$$

with an error of at most .000022.

Example 2. $f(x) = e^x$, $a=0$.

For $x<0$ the derivative of e^x (which is e^x) is positive and bounded by 1, so, by the corollary,

$$R_n(x,0) \leq \frac{|x|^{n+1}}{(n+1)!}, \quad x<0.$$

How large must we take n to guarantee that $R_n(-1,0)$ is at most .0001 in absolute value? We must have $\frac{1}{(n+1)!} < .0001$, i.e., $(n+1)! > 10,000$. The smallest such n is 7 ($7! = 5,760,\ 8! = 40320$). The 7^{th} Taylor polynomial of e^x is

$$1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!}.$$

Setting $x = -1$, then, we see that the number

$$1 - 1 + \frac{1}{2} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} \approx .367857$$

is within $1/40320 \approx .000025$ of $1/e$.

Example 3. $f(x) = \sqrt{1+x}$, $a = 0$.

Let's use the fourth Taylor polynomial at the origin to get an estimate for $\sqrt{3/2}$. We have

$$\begin{aligned}
f'(x) &= \frac{1}{2}(1+x)^{-1/2},\ f'(0) = \frac{1}{2} \\
f''(x) &= -\frac{1}{4}(1+x)^{-3/2},\ f''(0) = -\frac{1}{4} \\
f^{(3)}(x) &= \frac{3}{8}(x+1)^{-5/2},\ f^{(3)}(0) = \frac{3}{8} \\
f^{(4)}(x) &= -\frac{15}{16}(1+x)^{-7/2},\ f^{(4)}(0) = -\frac{15}{16}.
\end{aligned}$$

From this one finds that the fourth Taylor polynomial at the origin is given by

$$p_4(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{3}{48}x^3 - \frac{15}{384}x^4.$$

Our approximation to $\sqrt{3/2}$ is thus

$$p_4\left(\frac{1}{2}\right) = 1 + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) - \frac{1}{8}\left(\frac{1}{4}\right) + \frac{3}{48}\left(\frac{1}{8}\right) - \frac{15}{384}\left(\frac{1}{16}\right) \approx 1.22412.$$

To obtain an error estimate we note that

$$f^{(5)}(x) = \frac{105}{32}(1+x)^{-9/2},$$

which is bounded by $105/32$ for $x > 0$. By the corollary,

$$\left|R_4\left(\frac{1}{2}, 0\right)\right| \leq \frac{105}{32}\left(\frac{2^{-5}}{5!}\right) < .00086.$$

Our approximate value for $\sqrt{3/2}$ is accurate to within .00086.