Math 16B – F05 – Supplementary Notes 5
The Differential Notation

The differential notation is a convenient formalism which, once one becomes accustomed to it, can often provide a smooth way of carrying out integrations. The differential $df(x)$ of the function $f(x)$ is by definition the formal expression $f'(x)dx$. Here, the "dx" in the expression is not to be thought of as "d" times "x". It is rather "the differential of x," a formal expression. Frequently one notationally suppresses the dependence of the function f on the variable x and writes simply

$$df = f'dx. \tag{1}$$

In a formal sense, then, f' is the ratio of df by dx; you can imagine that f' is obtained by dividing through by dx in (1). The differential notation is thus an adjunct to the notation $\frac{df}{dx}$ for f'. (If you wish, you may interpret the symbol dx on the right side of (1) as a mechanism for keeping track of the variable x on which f depends.)

The notation goes back to G. Leibniz, one of the founders of calculus. Leibniz thought of dx as an "infinitesimal" change in the variable x and of df as the corresponding change in the function f. For him, the derivative f' was the actual ratio of the two "infinitesimals" df and dx. When calculus was put on a rigorous basis in the 19^{th} century the imprecise notion of an infinitesimal was discarded, replaced by the notion of a limit (although infinitesimals can still be helpful on an intuitive level). Leibniz's notation, however, is still with us.

Below are some of the differentiation formulas we have learned, expressed in the language of differentials.

$$\begin{aligned}
d(x^a) &= ax^{a-1}dx \\
d(e^x) &= e^x dx \\
d(\ln|x|) &= \frac{1}{x}dx \\
d(\sin x) &= \cos x\ dx \\
d(\cos x) &= -\sin x\ dx \\
d(\tan x) &= \sec^2 x\ dx \\
d(\sec x) &= \sec x \tan x\ dx
\end{aligned}$$

In differential notation, the sum rule, product rule, and quotient rule for differentiation read as follows.

$$\begin{aligned}
d(f+g) &= df + dg \\
d(fg) &= gdf + fdg \\
d\left(\frac{f}{g}\right) &= \frac{gdf - fdg}{g^2}.
\end{aligned}$$

The chain rule requires a bit of explanation. Suppose we have a composite function $f(u(x))$, i.e., f is a function of the variable u and u is a function of the variable x. We can then write df in two different ways, namely,

$$df = f'(u)du = f'(u(x))u'(x)dx. \tag{2}$$

Thus, for example,

$$d(e^{x^2}) = e^{x^2}\ d(x^2) = 2x\ e^{x^2}\ dx. \tag{3}$$

To compare (3) with (2) think of e^{x^2} as $e^{u(x)}$ where $u(x) = x^2$, and recall that $\frac{d}{du}(e^u) = e^u$.

In (2) we have been explicit about the dependencies among the variables f, u and x. Often, to unclutter the notation, one suppresses those dependencies. For example, in place of (2) one can write

$$df = \frac{df}{du}du = \frac{df}{du}\frac{du}{dx}dx.$$

When using this shorthand notation, one must take care not to lose track of which variables depend on which others.

The Differential Notation and Integration

The fundamental theorem of calculus, in one of its versions, reads

$$\int f'(x)dx = f(x) + C.$$

Translated into differential notation, this becomes

$$\int df = f + C. \tag{4}$$

Integration by Substitution. The method of substitution can be used when an integrand has the form $f'(u(x))u'(x)$. Using the differential notation, we can write

$$\int f'(u(x))u'(x)dx = \int df(u).$$

Accordingly, by (4),

$$\int f'(u(x))u'(x)dx = f(u) + C,$$

or, more explicitly,

$$\int f'(u(x))u'(x)dx = f(u(x)) + C.$$

The key to using the method lies in recognizing when a given integrand has the appropriate form.

Example 1. $\int x\ e^{x^2}\ dx$.

We have (compare with (3))

$$\int x\ e^{x^2}\ dx = \frac{1}{2}\int e^{x^2}\ d(x^2) = \frac{1}{2}e^{x^2} + C.$$

Here, we have recognized that, except for the factor $\frac{1}{2}$, the integrand can be written as $e^{u(x)}$ with $u(x) = x^2$. We then, without writing it out explicitly, used the formula $\int e^u\ du = e^u + C$. After a little practice one is often able to make the appropriate substitution "in one's head," as in this example.

Example 2. $\int \tan x\ \sec^2\ x\ dx$.

Since $\frac{d}{dx}(\tan x) = \sec^2 x$, we have

$$\int \tan x \sec^2 x \, dx = \int \tan x \, d(\tan x) = \frac{\tan^2 x}{2} + C.$$

Example 3. $\int \sin x \cos^3 x \, dx$.

We'll perform this integration in two ways. First,

$$\int \sin x \cos^3 x \, dx = -\int \cos^3 x \, d(\cos x) = -\frac{\cos^4 x}{4} + C. \tag{5}$$

On the other hand, since $\cos^2 x = 1 - \sin^2 x$, we have

$$\begin{aligned} \int \sin x \cos^3 x \, dx &= \int (\sin x - \sin^3 x) \cos x \, dx \\ &= \int (\sin x - \sin^3 x) d(\sin x) = \frac{\sin^2 x}{2} - \frac{\sin^4 x}{4} + C. \end{aligned} \tag{6}$$

To reconcile the right sides of (5) and (6) we note that

$$\frac{\cos^4 x}{4} = \frac{(1 - \sin^2 x)^2}{4} = \frac{1}{4} - \frac{\sin^2 x}{2} + \frac{\sin^4 x}{4}.$$

The functions $-\frac{\cos^4 x}{4}$ and $\frac{\sin^2 x}{2} - \frac{\sin^4 x}{4}$ on the right sides of (5) and (6) thus differ by the constant $-\frac{1}{4}$. The right sides of (5) and (6) are accordingly equivalent, because the "C" appearing in each expression represents an *arbitrary* constant. (Remember that an indefinite integration produces a family of functions, not a single function.)

Example 4. $\int \cot x \, dx$.

We have

$$\begin{aligned} \int \cot x \, dx &= \int \frac{\cos x}{\sin x} dx = \int \frac{1}{\sin x} d(\sin x) \\ &= \ln|\sin x| + C. \end{aligned}$$

Integration by Parts. The differential version of the product rule, $d(fg) = gdf + fdg$, can be rewritten as

$$fdg = d(fg) - gdf.$$

It follows that

$$\int fdg = fg - \int gdf, \tag{7}$$

which is the formula for integration by parts. The method is useful when we want to perform the integration $\int fdg$ and already know how to perform the integration $\int gdf$. As with the method of substitution, the key to using the method of integration by parts lies in recognizing when a given integrand is of the appropriate form. (One must occasionally resort to trial and error.)

Example 1. $\int \frac{\ln x}{x^2} dx$.

We have

$$\begin{aligned}
\int \frac{\ln\, x}{x^2} dx &= -\int \ln\, x \; d\left(\frac{1}{x}\right) \\
&= -\frac{1}{x}\ \ln\, x + \int \frac{1}{x} d(\ln\, x) \\
&= -\frac{1}{x}\ \ln\, x + \int \frac{1}{x^2} dx \\
&= -\frac{1}{x}\ \ln\, x - \frac{1}{x} + C.
\end{aligned}$$

Here, without being explicit, we have used (7) with $f(x) = -\ln\, x$ and $g(x) = \frac{1}{x}$.

Example 2. $\int x^2\ e^{-x}\ dx$.

Here we integrate by parts twice:

$$\begin{aligned}
\int x^2\ e^{-x}\ dx &= -\int x^2\ d(e^{-x}) \\
&= -x^2\ e^{-x} + \int e^{-x}\ d(x^2) \\
&= -x^2\ e^{-x} + 2\int x\ e^{-x}\ dx \\
&= -x^2\ e^{-x} - 2\int x\ d(e^{-x}) \\
&= -x^2\ e^{-x} - 2x\ e^{-x} + 2\int e^{-x}\ dx \\
&= -x^2\ e^{-x} - 2x\ e^{-x} + 2e^{-x} + C.
\end{aligned}$$

Example 3. $\int x^2 \cos x\ dx$.

This is similar to the preceding example:

$$\begin{aligned}
\int x^2 \cos x\ dx &= \int x^2\ d(\sin x) \\
&= x^2 \sin x - \int \sin x\ d(x^2) \\
&= x^2 \sin x - 2\int x\ \sin x\ dx \\
&= x^2 \sin x + 2\int x\ d(\cos x) \\
&= x^2 \sin x + 2x\cos x - 2\int \cos x\ dx \\
&= x^2 \sin x + 2x\cos x - 2\sin x + C.
\end{aligned}$$