Math 16B – F05 – Supplementary Notes 4
The Derivatives of $\sin t$ and $\cos t$

One can derive the formulas

$$\frac{d}{dt}(\sin t) = \cos t, \quad \frac{d}{dt}(\cos t) = -\sin t \tag{1}$$

starting from the relations

$$\lim_{t\to 0} \frac{\sin t}{t} = 1 \tag{2}$$

$$\lim_{t\to 0} \frac{\cos t - 1}{t} = 0. \tag{3}$$

Note that (2) and (3) just say that (1) holds at the origin (since $\cos 0 = 1$ and $\sin 0 = 0$).

Taking (2) and (3) temporarily for granted, let's derive (1). By definition of the derivative,

$$\frac{d}{dt}(\sin t) = \lim_{h\to 0} \frac{\sin(t+h) - \sin t}{h}.$$

We use the addition formula $\sin(t+h) = \sin t \cos h + \cos t \sin h$ to rewrite this as

$$\frac{d}{dt}(\sin t) = \lim_{h\to 0} \left[\sin t \left(\frac{\cos h - 1}{h}\right) + \cos t \left(\frac{\sin h}{h}\right)\right].$$

By (2) and (3) the limit on the right side equals $\cos t$, which establishes the first formula in (1). The second formula in (1) can be deduced from the first one by means of the identities

$$\cos t = \sin\left(t + \frac{\pi}{2}\right), \quad \sin t = -\cos\left(t + \frac{\pi}{2}\right)$$

and the chain rule. We have

$$\begin{aligned} \frac{d}{dt}(\cos t) &= \frac{d}{dt}\left(\sin\left(t + \frac{\pi}{2}\right)\right) = \cos\left(t + \frac{\pi}{2}\right) \frac{d}{dt}\left(t + \frac{\pi}{2}\right) \\ &= \cos\left(t + \frac{\pi}{2}\right) = -\sin t. \end{aligned}$$

So, to establish (1), it only remains to establish (2) and (3). Once (2) is known (3) follows easily. In fact,

$$\begin{aligned} \frac{\cos t - 1}{t} &= \frac{(\cos t - 1)(\cos t + 1)}{t(\cos t + 1)} = \frac{\cos^2 t - 1}{t(\cos t + 1)} \\ &= \frac{-\sin^2 t}{t(\cos t + 1)} = -\sin t \left(\frac{\sin t}{t}\right)\left(\frac{1}{\cos t + 1}\right). \end{aligned}$$

As t tends to 0, the first factor on the right side tends to 0 (since $\sin 0 = 0$) and the last factor tends to $\frac{1}{2}$ (since $\cos 0 = 1$). By (2), the middle factor tends to 1, so the product tends to $(0)(1)\left(\frac{1}{2}\right) = 0$, which gives (3).

The relation (2) is thus the basic one. We'll derive it using some simple geometry. Let t be a small positive angle. (Since $\frac{\sin t}{t}$ is an even function of t, it suffices to establish (2) as t tends to

0 through positive values.) From the point $A = (\cos t, \sin t)$ on the unit circle we construct the tangent line to the circle, and we let B denote the point where the tangent line intersects the x-axis (see Figure 4.1). The origin will be denoted by O.

The distance of the point A from the x-axis is $\sin t$, which is less than the length of the arc of the unit circle subtended by the angle t. The preceding arc has length t (by the definition of radians), so we have the inequality $\sin t < t$, which we can write as $\frac{\sin t}{t} < 1$.

To obtain a lower bound for $\frac{\sin t}{t}$ we consider the right triangle OAB. The side adjacent to the angle t has length 1, so the side opposite the angle t has length $\tan t$. The area of the triangle is therefore $\frac{1}{2}(1)(\tan t) = \frac{\tan t}{2}$. The triangle contains the sector of the unit circle cut off by the angle t, so its area is larger than the area of the sector. The area of the sector equals the area of the whole circle, which is π, times $\frac{t}{2\pi}$, the ratio of t to the length of the full circle. The area of the sector is thus $\frac{t}{2}$, giving us the inequality $\frac{\tan t}{2} > \frac{t}{2}$, which we can rewrite as $\frac{\sin t}{t} > \cos t$.

We now have the pair of inequalities

$$\cos t < \frac{\sin t}{t} < 1.$$

Since $\lim_{t \to 0}(\cos t) = 1$, the relation (2) follows.