Math 16B – F05 – Supplementary Notes 2
Second-Derivative Test

To understand what is behind the second-derivative test for functions of two variables, we shall start by looking at the simplest nontrivial example, that of a polynomial of degree 2. First the test will be stated.

Let the function $f(x, y)$ have a critical point at (a, b). The second-derivative test involves the function

$$D_f(x, y) = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2,$$

and it applies when $D_f(a, b) \neq 0$. Note that if $D_f(a, b) > 0$ then $\frac{\partial^2 f}{\partial x^2}(a, b)$ and $\frac{\partial^2 f}{\partial y^2}(a, b)$ must have the same sign. The test distinguishes three cases:

(I) If $D_f(a, b) > 0$ and $\frac{\partial^2 f}{\partial x^2}(a, b) > 0$ (equivalently $\frac{\partial^2 f}{\partial y^2}(a, b) > 0$), then (a, b) is a relative minimum of f.

(II) If $D_f(a, b) > 0$ and $\frac{\partial^2 f}{\partial x^2}(a, b) < 0$ (equivalently $\frac{\partial^2 f}{\partial y^2}(a, b) < 0$), then (a, b) is a relative maximum of f.

(III) If $D_f(a, b) < 0$ then (a, b) is a saddle point of f (neither a relative maximum nor a relative minimum).

Now we look at the simple example $f(x, y) = \alpha x^2 + 2\beta xy + \gamma y^2$, where α, β, γ are constants, not all 0. This quadratic polynomial has a critical point at $(0, 0)$, where it takes the value 0. We have

$$\frac{\partial^2 f}{\partial x^2} = 2\alpha, \quad \frac{\partial^2 f}{\partial y^2} = 2\gamma, \quad \frac{\partial^2 f}{\partial x \partial y} = 2\beta,$$
$$D_f(x, y) = 4(\alpha\gamma - \beta^2).$$

The second-derivative test can be derived for this function f by means of elementary algebra. To illustrate, the case $\alpha > 0$ will be discussed; the other cases are similar (except for the case $\alpha = \gamma = 0 \neq \beta$, which is simpler, in fact trivial). Assuming $\alpha > 0$, we use the method of completing the square. Namely, we look at $\alpha x^2 + 2\beta xy$, the first two terms in the expression for f, and determine what expression of the form δy^2 we can add to it to produce a perfect square. After a little thought one discovers that $\delta = \beta^2/\alpha$ works:

$$\alpha x^2 + 2\beta xy + \frac{\beta^2}{\alpha} y^2 = \left(\sqrt{\alpha}x + \frac{\beta}{\sqrt{\alpha}} y\right)^2.$$

We can thus rewrite f as

$$f(x, y) = \left(\sqrt{\alpha}x + \frac{\beta}{\sqrt{\alpha}} y\right)^2 + \frac{\alpha\gamma - \beta^2}{\alpha} y^2. \tag{1}$$

Now let's examine what happens in case $D_f(0, 0) = 4(\alpha\gamma - \beta^2)$ is positive, negative, or zero.

(i) If $D_f(0, 0) > 0$ then the coefficient multiplying y^2 in (1) is positive, and f is everywhere positive except at $(0, 0)$, which is thus a relative minimum (in agreement with the test).

(ii) If $D_f(0,0) < 0$ then the coefficient multiplying y^2 in (1) is negative. On the y axis the function f is positive except at $(0,0)$, but on the line $\sqrt{\alpha}x + \frac{\beta}{\sqrt{\alpha}}y = 0$ it is negative except at $(0,0)$. The critical point $(0,0)$ is a saddle point (again, in agreement with the test).

(iii) If $D_f(0,0) = 0$ then the y^2-term in (1) drops out. The function f is nonnegative and vanishes everywhere on the line $\sqrt{\alpha}x + \frac{\beta}{\sqrt{\alpha}}y = 0$, every point of which is thus a relative minimum. (This case is not covered by the test.)

The preceding discussion illustrates the fact that for the function $f(x,y) = \alpha x^2 + 2\beta xy + \gamma y^2$ one can derive the second-derivative test by purely algebraic means. It will now be indicated how the general case can be deduced from this special one by an approximation argument.

We return to the general function $f(x,y)$. The local linear approximation of f near a point (a,b) was discussed in Supplementary Notes 1. The approximation is expressed by

$$f(x,y) \approx f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b). \tag{2}$$

The expression (2) is shorthand for a more precise statement that indicates how the error in the approximation behaves as (x,y) approaches (a,b).

The approximation (2) is called a first-order approximation because it is an approximation of f by a first-degree polynomial. There are analogous, more accurate, higher-order approximations. The second-order approximation, which approximates f near (a,b) by a second-degree polynomial, reads

$$\begin{aligned} f(x,y) \approx\ & f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b) \\ & + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(a,b)(x-a)^2 + \frac{\partial^2 f}{\partial x \partial y}(a,b)(x-a)(y-b) + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(a,b)(y-b)^2, \end{aligned} \tag{3}$$

and there is an associated error estimate.

Now suppose (a,b) is a critical point of f, and let

$$\alpha = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(a,b), \quad \beta = \frac{1}{2}\frac{\partial^2 f}{\partial x \partial y}(a,b), \quad \gamma = \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(a,b).$$

Then (3) reduces to

$$f(x,y) \approx f(a,b) + \alpha(x-a)^2 + 2\beta(x-a)(y-b) + \gamma(y-b)^2. \tag{4}$$

The second-degree polynomial on the right side here is a trivially modified version of the one discussed earlier; it has a critical point at (a,b) instead of at $(0,0)$, and it may not vanish at the critical point, but those changes are merely superficial. One can verify the second-derivative test for the polynomial on the right side of (4) by purely algebraic means.

Finally, using the error estimate that accompanies (4), one can show that, as long as $D_f(a,b)$ $(= 4(\alpha\gamma - \beta^2))$ is not zero, the nature of (a,b) as a critical point of f is controlled by the nature of (a,b) as a critical point of the polynomial on the right side of (4). This is how the second-derivative test is established.