Math 16B, Section 1 **Fall 2005**
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REVIEW EXERCISES 3

1. Find and classify (by means of the second-derivative test) the critical points of the functions f below.
   (a) $f(x,y) = 8x^3 + y^3 - xy$ (b) $f(x,y) = 5x^2 + y^2 - x - y - 2xy$

2. Which linear function $ax + b$ best approximates the function x^2 at $x = 0, 1, 2$ in terms of least squares?

3. Maximize the function $f(x,y) = 27x^{1/3}y^{2/3}$ in the region $x > 0$, $y > 0$, under the constraint $x^2 + 2y^2 = 12$.

4. Evaluate the integrals.
   (a) $\iint_R x^2 y\,dx\,dy$, where R is the triangle with vertices $(-1,0)$, $(0,1)$, $(1,0)$.
   (b) $\iint_R (x^2 - y^2)\,dx\,dy$, where R is the region $0 \le y \le \sqrt{x}$, $0 \le x \le 1$.
   (c) $\int_0^1 x^3(x^2+1)^{1/2}\,dx$.
   (d) $\int_0^{\pi^2} \sin\sqrt{x}\,dx$.
   (e) $\int_1^e (\ln x)^3\,dx$.

5. For the differential equation $yy' = (y^2 - 1)t$:
   (a) What are the constant solutions?
   (b) What is the general solution?
   (c) What is the solution satisfying $y(0) = 2$?
   (d) What is the solution satisfying $y(0) = -2$?

6. Solve the initial-value problems.
   (a) $(y-1)^2(t^2+1)^4 y' = -2t$, $y(0) = 1$.
   (b) $y' = y\cos t$, $y(0) = -1$.
   (c) $y' = y + t^2$, $y(0) = 0$.
   (d) $y' = -y^3t^3$, $y(0) = -1$.
   (e) $y' = -y^3t^3$, $y(0) = 0$.

7. Horace and Hortense Snootee set up a college fund for their son Rudy on the day he is born. They start the fund with a \$20,000 deposit and will add \$7,000 per year. The fund is expected to grow at the rate of 7% per year, compounded continuously.

(a) Assuming the Snootee's yearly additions to the fund are made continuously, set up a differential equation satisfied by the amount $P(t)$ in the fund t years after its inception.

(b) Find the general solution of the differential equation.

(c) Find the solution of the differential equation satisfying the initial condition $P(0) = 20,000$.

(d) Determine the balance in the fund on Rudy's 18th birthday.

8. In the kingdom of Xerotopia on the planet Nullphoria, the birth rate is 13 births per thousand inhabitants per year and the death rate is 14 deaths per thousand inhabitants per year. Suppose the kingdom gains inhabitants through immigration at the constant rate of A inhabitants per year.

(a) Assuming as an approximation that births, deaths and immigration occur continuously, set up a differential equation satisfied by the population $P(t)$ of Xerotopia at time t.

(b) Find the general solution of the differential equation.

(c) Suppose the present population of Xerotopia is 40,000,000. How large must A be to guarantee that the future population will not fall below that number?

9. For the function $f(x) = \tan x$, find the second and third Taylor polynomials, $p_2(x)$ and $p_3(x)$, at $x = 0$.

10. (a) For the function $f(x) = \cos x$, find the fourth and fifth Taylor polynomials, $p_4(x)$ and $p_5(x)$, at $x = 0$.

(b) Use $p_4(x)$ to estimate $\cos \dfrac{1}{2}$, and use the remainder etimate to bound the error.

11. (a) For the function $f(x) = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6}$, determine the third Taylor polynomial $p_3(x)$ at $x = 1$.

(b) For the same function, determine the third Taylor polynomial at $x = 0$.

(c) In what way do the polynomials found in parts (a) and (b) differ?

12. (a) Write the repeating decimal $.909090\ldots$ as a fraction.

(b) Find the third Taylor polynomial $p_3(x)$ for the function $f(x) = \dfrac{1}{x}$ at $x = 1$.

(c) Use $p_3(x)$ from part (b) to estimate $\dfrac{1}{1.1}$. Bound the error using the remainder estimate.

(d) Explain the connection between parts (a) and (c).

13. (a) Find the Taylor series at $x = 0$ for the function $f(x) = 1/(1-x)^3$.

(b) Use the result from (a) to evaluate the sum $\displaystyle\sum_{n=0}^{\infty}(n+1)(n+2)2^{-n}$.

14. (a) Find the second Taylor polynomial $p_2(x)$ at $x = 100$ for the function $f(x) = \sqrt{x}$.

(b) Use the result from (a) to estimate $\sqrt{104}$. Use the remainder estimate to bound the error.

15. A fair coin is tossed three times in succession. The number of heads that come up is a discrete random variable X whose possible values are $0, 1, 2, 3$. The associated probabilities are $1/8, 3/8, 3/8, 1/8$, respectively. Calculate

    (a) the expected value $E(X)$.

    (b) the variance $\mathrm{Var}(X)$.

    (c) the standard deviation of X.

16. Compute the expected value $E(X)$ and the variance $\mathrm{Var}(X)$ of a continuous random variable X whose probability density function is the function $f(x) = \dfrac{1}{2}\sin x$, $0 \le x \le \pi$.

17. (a) Shifty Shafter's weighted coin has the probability $51/100$ of coming up heads. If tossed three times in succession, a fair coin has the probability $1/8 = .125$ of coming up heads each time, and the same probability of coming up tails each time. Compute the corresponding probabilities for Shifty's coin.

    (b) Suppose Shifty bets 7 of his dollars against 1 of yours that a head will appear at least once in three successive tosses of his coin. The amount Shifty wins is then a random variable X whose possible values are 1 and -7. Compute the expected value of X.