MATH 113 Homework 1: Sets and mappings
SOLUTIONS

NOTE: These solutions are not unique!

1. In class we proved a distributive property for set unions and intersections. Now describe an associative property for set intersections and prove this property.

   SOLUTION: $(A \cap B) \cap C = A \cap (B \cap C)$.

   Step 1: Show $(A \cap B) \cap C \subseteq A \cap (B \cap C)$.

   $x \in (A \cap B) \cap C \Rightarrow x \in A \cap B$ and $x \in C$. $x \in A \cap B \Rightarrow x \in A$ and $x \in B$. So $x \in A$ and $x \in B$ and $x \in C$. So $x \in A$ and $x \in B \cap C$. Therefore $x \in A \cap (B \cap C)$.

   Step 2: Show $A \cap (B \cap C) \subseteq (A \cap B) \cap C$.

   $x \in A \cap (B \cap C) \Rightarrow x \in A$ and $x \in B \cap C$. $x \in B \cap C \Rightarrow x \in B$ and $x \in C$. So $x \in A$ and $x \in B$ and $x \in C$. So $x \in A \cap B$ and $x \in C$. Therefore $x \in (A \cap B) \cap C$.

2. Is set containment an equivalence relation? (Explain your answer.)

   SOLUTION: No

   It does not have the symmetric property.

   If A is a proper subset of B, then B is not a proper subset of A. For example, let $A = \{1, 2, 3\}$ and let $B = \{1, 2, 3, 4\}$. Then $A \subseteq B$ but $B \nsubseteq A$ since $4 \notin A$.

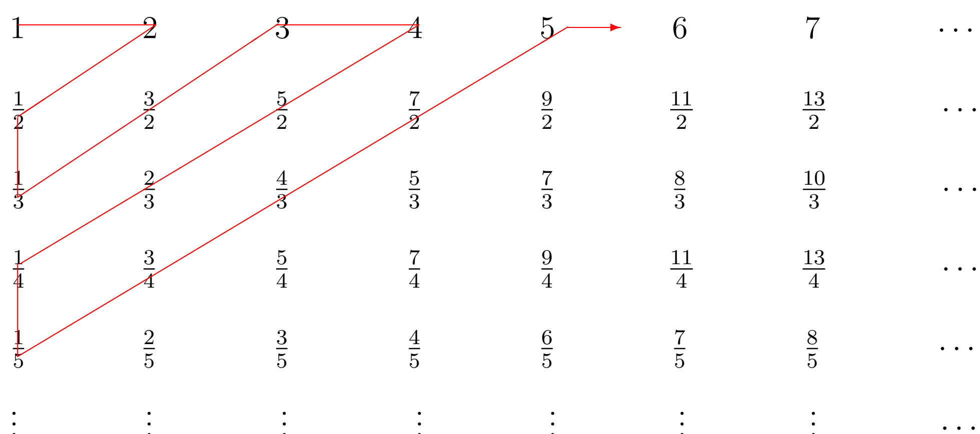
3. If $a \equiv 2 \bmod 4$ and $b \equiv 3 \bmod 4$, then ab is congruent to what mod 4? (Explain your answer.)

   SOLUTION: ab is congruent to 2 mod 4.

Let $a = 4k + 2$ for some k and $b = 4j + 3$ for some j. Then $ab = (4k+2)(4j+3) = 16k^2 + 12k + 8j + 6 = 4(4k^2 + 3k + 2j + 1) + 2$. So $ab \equiv 2 \bmod 4$.

4. Does a bijection from $\mathbb{Q}^+$ to $\mathbb{Z}$ exist? If so, describe it. If not, why not?

   SOLUTION: If we record the elements of $\mathbb{Q}^+$ as an infinite array (similar to the array for $\mathbb{Q}$ on p.5 of the text), writing only lowest terms fractions, we get the following picture:



   Here the red arrow describes a function f. The red arrow begins at the element of $\mathbb{Q}^+$ which maps to 0 under f. The second element encountered by the arrow is the element of $\mathbb{Q}^+$ which maps to 1 under f. The third element encountered by the arrow is the element of $\mathbb{Q}^+$ which maps to -1 under f. Continuing this way, the i^{th} elemented encountered by the arrow maps to the element $\frac{i}{2}$ of $\mathbb{Z}$ is i is even and to the element $\frac{1-i}{2} \in \mathbb{Z}$ is i is odd.

   <u>f is injective:</u>

   Let $f(a) = f(b) = z \in \mathbb{Z}$. If z is non-negative then a is the $2z^{th}$ element encountered by the red arrow, as is b. So $a = b$. Similarly, if z is negative then a is the $(1-2z)^{th}$ element of the array encountered by the red arrow, as is b. So $a = b$. Therefore f is injective.

   <u>f is surjective:</u>

Let $z \in \mathbb{Z}$. If z is non-negative then let a be the $2z^{th}$ element encountered by the red arrow. Then $f(a) = z$. If z is negative then let a be the $(1-2z)^{th}$ element encountered by the red arrow. Then $f(a) = z$. Therefore for each $z \in \mathbb{Z}$, there exists an element $a \in \mathbb{Q}^+$ such that $f(a) = z$, which means that f is surjective.

What about an injection which is not surjective?

SOLUTION: Use the same picture but let g be the map which sends the first element encountered by the red arrow to 1, the second element encountered by the arrow to 2, and so on. The map g is injective by the same argument as above but it is not surjective since the negative integers (and 0) are not in the image $g(\mathbb{Q}^+)$.

A surjection which is not injective?

SOLUTION: Define a function h by

$$h(\tfrac{a}{b}) = \begin{cases} a & \text{if } a < b \\ -b & \text{if } a > b \\ 0 & \text{if } a = b \end{cases}.$$

h is surjective:

Let $z \in \mathbb{Z}$. If $z > 0$, then $z = h(\frac{z}{z+1})$. If $z < 0$, then $z = h(\frac{|b|+1}{|b|})$. If $z = 0$, then $z = g(\frac{1}{1})$. So for each $z \in \mathbb{Z}$, there exists an $a \in \mathbb{Q}^+$ such that $h(a) = z$. So h is surjective.

h is not injective:

Let $a = \frac{3}{5}$ and $b = \frac{3}{7}$. Then $h(a) = h(b) = 3$. So h is not injective.

Group Problem (Presentation Tuesday, September 5)

Let S be a set, and let $\mathcal{P}(S)$ be the set of all subsets of S, called the **power set** of S.

1. Can you find a bijection from $\mathcal{P}(\mathbb{Z})$ to a set we talked about in class?

   SOLUTION:

   The answer is yes, there is a bijection from $\mathcal{P}(\mathbb{Z})$ to the real numbers $\mathbb{R}^+$.

Let $S \in \mathcal{P}(\mathbb{Z})$. Then S maps to an infinite sequence of zeros and ones. This can be seen by letting the z^{th} entry of the sequence be 1 if $z \in S$ and letting the z^{th} entry of the sequence be 0 otherwise. For instance, the set $S = \{-5, -2, 1, 2, 4\}$ maps to the sequence $s = \ldots 0, 0, 0, 0, 1, 0, 0, 1, 0, \underline{0}, 1, 1, 0, 1, 0, 0, 0, 0, 0 \ldots$, where the underlined zero corresponds to the 0^{th} entry in the sequence.

This sequence can be mapped to a real number by letting

$$f(s) = \sum_{k=-\infty}^{\infty} s_k 2^k,$$

where s_k is the k^{th} entry in the sequence s.

In other words, the sequence s is a binary representation of $f(s)$.

f is surjective:

Let $r \in \mathbb{R}^+$. Then r has a binary representation of the form above. (The part of r to the left of the decimal is given by the non-negative part of the sequence and the part to the right of the decimal is given by the negative part.)

f is (almost) injective:

Assume $f(s) = f(t)$. Then

$$\sum_{k=-\infty}^{\infty} s_k 2^k = \sum_{k=-\infty}^{\infty} t_k 2^k.$$

The problem is that for some real numbers there are actually two different sequences s and t that satisfy this. For example, $3 = 2.999999999\ldots$. So the sequences $s = \ldots 0, 0, 0, 0, \underline{1}, 1, 0, 0, 0, 0, \ldots$ and $t = \ldots 1, 1, 1, \underline{0}, 1, 0, 0, 0, \ldots$ give the same positive real number 3.

(If you are looking for a challenge, what can you do to fix this?)

2. Prove that for any set S (finite or infinite) it is impossible to find a mapping of S **onto** $\mathcal{P}(S)$.

   SOLUTION:

   Argue by contradiction. Assume that there exists a mapping f of S onto $\mathcal{P}(S)$. Define the set $A \in \mathcal{P}(S)$ by $A = \{x \in S | x \notin f(x)\}$. Since

f is surjective, there must be an element $y \in S$ such that $f(y) = A$. If $y \in f(y)$, then $y \in A$. But by the definition of A, $y \notin f(y)$. So we must have $y \notin f(y)$. But then $y \in A = f(y)$ by the definition of A. This contradicts our assumption that $y \notin f(y)$. Therefore no mapping from S to $\mathcal{P}(S)$ is surjective.