MATH 113: HW 14: Extra Credit

Due by 5pm Thursday, December 14. (Will replace lowest homework grade)

1. (10 pts) Section 29, Exercise 8 (prove that your polynomial is irreducible).

   SOLUTION:

   (8) $\alpha = \sqrt{2}+i$. Then $\alpha^2 = 2+2\sqrt{2}i-1 = 1+2\sqrt{2}i$. So $\alpha^2-1 = 2\sqrt{2}i$. Therefore $(\alpha^2-1)^2 = 8(-1)$. So $\alpha^4 - 2\alpha^2 + 9 = 0$. Let $f(x) = x^4 - 2x^2 + 9$. Then $f(x) \in \mathbb{Z}[x]$. If $f(x)$ has a zero in $\mathbb{Q}$, then $f(x)$ has a zero in $\mathbb{Z}$ which divides 9. So $f(x)$ does not have a zero in $\mathbb{Q}$. Therefore if $f(x)$ is reducible, then it factors into two quadratic factors:

   $$f(x) = (x^2 + ax + b)(x^2 + cx + d),$$

   in $\mathbb{Z}[x]$.

   Equating coefficients of powers of x, we see that

   $$a + c = 0, \quad b + d + ac = -2, \quad ad + bc = 0, \quad \text{and} \quad bd = 3$$

   Then b and d are either 1 and 3 or -1 and -3. This implies that either $-4 - a^2 = -2$ or $4 - a^2 = -2$. Thus $a^2 = -2$, which cannot occur. Therefore $f(x)$ is irreducible.

   So $irr(\alpha, \mathbb{Q}) = x^4 - 2x^2 + 3$ and $deg(\alpha, \mathbb{Q}) = 4$.

2. (20 pts)

   (a) If $a \in K$ is transcendental over F, let $F(a) = \{\frac{f(a)}{g(a)} | f(x), g(x) \neq 0 \in F[x]\}$. Show that $F(a)$ is a field and is the smallest subfield of K containing both F and a.

   SOLUTION:

   Let $F[a] = \{f(a) | f(x) \in F[x]\}$. Then $F[a]$ is the image of $F[x]$ under the evaluation homomorphism ϕ_a. Therefore $F[a]$ is a subring of K. Its quotient field is given by $F(a)$, so $F(a)$ is the smallest field containing $F[a]$. Any field containing both F and a must contain $F[a]$, so $F(a)$ is the smallest subfield of K containing both F and a.

   (b) Show that $F(a) \cong F(x)$, where $F(x)$ is the field of rational functions in x over F.

SOLUTION:

Let $\phi_a : F(x) \rightarrow F(a)$ be the evaluation homomorphism. We know this is a homomorphism, so we must prove it is injective and surjective.

ϕ_a is injective: Assume that $\phi_a(\frac{f(x)}{g(x)}) = 0$ for some $f(x), g(x)$ where $g(x) \neq 0$. Then $\frac{f(a)}{g(a)} = 0$. Multiply both sides by $g(a)$ to see that $f(a) = 0$. But a is transcendental, so this cannot occur. So ϕ_a is injective.

ϕ_a is surjective: Let $\frac{f(a)}{g(a)} \in F(a)$. Then $\frac{f(a)}{g(a)} = \phi_a(\frac{f(x)}{g(x)})$. So ϕ_a is surjective.

So $F(x)$ and $F(a)$ are isomorphic.

3. (10 pts) Prove that the number of transcendental numbers is uncountable.

SOLUTION:

Let A be the set of algebraic numbers and let T be the set of transcendental numbers. Then $\mathbb{C} = A \cup T$. If A is countable, then T must be uncountable, since the union of two countable sets is countable. Therefore we will prove that A is countable.

Every algebraic number must be the zero of a polynomial with coefficients in $\mathbb{Q}$. Consider the set of polynomials of degree n in $\mathbb{Q}[x]$. There are $n+1$ coefficients in each such polynomial, so the set C_n of all such polynomials is isomorphic to $\mathbb{Q}^{n+1}$. (Map the i^{th} coefficient to the i^{th} coordinate.) This set is countable, since a finite product of countable sets is countable. (Use an $(n+1)$-dimensional version of the argument used to prove $\mathbb{Z} \times \mathbb{Z}$ is countable.)

The set of all polynomials is isomorphic to the union $\cup_{n=0}^{\infty} C_n$ of the above sets. A countable union of countable sets is countable, so this union is countable. Since this is an upper bound on the number of algebraic numbers and there are infinitely many algebraic numbers, the set of algebraic numbers is countable. Therefore T is uncountable.

4. (10 pts) Section 31, Exercise 6 OR Section 29, Exercises 14 and 16.

SOLUTION:

(31.6) Let $\alpha = \sqrt{2} + \sqrt{3}$. Then $\alpha^2 = 2 + 2\sqrt{6} + 3 = 5 + 2\sqrt{6}$. So $\alpha^2 - 5 = 2\sqrt{6}$. Then $(\alpha^2 - 5)^2 = 24$, which means that $\alpha^4 - 10\alpha^2 + 25 = 24$. So $\alpha^4 - 10\alpha^2 + 1 = 0$. Therefore α is a zero of the polynomial $f(x) = x^4 - 10x^2 + 1$.

If $f(x)$ is reducible and $f(x)$ has a linear factor, then by Corollary 23.3 $f(x)$ has a zero in $\mathbb{Z}$. By corollary 23.12, such a zero must divide the constant term, 1, of $f(x)$. But $f(1) = -8$ and $f(-1) = -8$. So there are no linear factors. Therefore if $f(x)$ is reducible, it factors into two quadratic polynomials, $p(x) = a_2x^2 + a_1x + a_0$ and $q(x) = b_2x^2 + b_1x + b_0$. Then $a_1b_0 + a_0b_1 = 0$. Since $a_0 = b_0 = 1 \text{ or } -1$, we have $a_1 + b_1 = 0$. We also know that $a_2b_2 = 1$, so $a_2 = b_2 = 1 \text{ or } -1$. Continuing such arguments eventually leads to a contradiction and we see that $f(x)$ is irreducible.

Since $f(x)$ is a monic irreducible polynomial, it is the polynomial $irr(\alpha, \mathbb{Q})$. Therefore the degree of $\mathbb{Q}(\sqrt{2} + \sqrt{3})$ equals the degree, 4, of $f(x)$. The basis for $\mathbb{Q}(\sqrt{2} + \sqrt{3})$ is $\{1, \alpha, \alpha^2, \alpha^3\}$.

(29.14) If $\alpha = \pi^2$ were algebraic over $\mathbb{Q}$, then $f(\pi^2) = 0$ for some polynomial $f(x) \in \mathbb{Q}[x]$. So $f(x^2)$ is a polynomial in $\mathbb{Q}[x]$ which has π as a zero. But π is transcendental, so this this is a contradiction.

(29.16) Consider the polynomial $f(x) = x^3 - (\pi^3)^2 \in \mathbb{Q}(\pi^3)$. The element π^2 is a zero of this polynomial, so π^2 is algebraic over $\mathbb{Q}(\pi^3)$.

5. (10 pts) If R is a commutative ring that is algebraically closed, is R necessarily a field? (Proof or counterexample.)

SOLUTION:

If R does not have to contain unity, then the ring $\{0\}$ is algebraically closed and we do not consider $\{0\}$ to be a field since it has no multiplicative identity.

If R does contain unity, then we need to prove that every element of R is a unit. Let a be an arbitrary element in R. Then the polynomial $f(x) = ax - 1$ must have a zero in R. Let α be the element of R such that $f(\alpha) = 0$. Then $a\alpha - 1 = 0$. So $a\alpha = 1$. So α is the multiplicative inverse of a. Therefore R is indeed a field.