MATH 113 (Section 2) Midterm 2

2 November 2006

Name: ____

You have 80 minutes to complete the 6 problems on this exam. The exam is scored out of 100 possible points, but some problems are worth more than others.

You may quote theorems and definitions from our textbook, class notes, and from basic linear algebra, but not from other upper division mathematics courses.

You may not use any materials such as books, notes, formula sheets, or calculators. No talking during the exam. Any academic honesty violations will be punished severely according to university guidelines.

Problem 1. **(14 pts)** Let $\sigma = (1,3)(2)(4,8,7,9)(5,10,14,11)(6,13,12)$ be a permutation in S_{14}.

(a) Is σ an even or odd permutation?

SOLUTION:

The sign of an odd length cycle is even and the sign of an even length cycle is odd. The permutation σ contains 3 cycles of even length, so there are 3 cycles with an odd number of transpositions. Therefore σ is an odd permutation.

(b) What is the order of σ in S_{14}?

SOLUTION:

The order of σ is the least common multiple of its cycle lengths. Therefore the order of σ is $lcm(2,1,4,4,3) = 12$.

Problem 2. (15 pts) Let X be a G-set and define
$G_X = \{g \in G \mid gx = x \quad \forall x \in X\}$. Prove G_X is a normal subgroup of G.

SOLUTION:

First prove that G_X is a subgroup of G.

Closure: Let $g_1, g_2 \in G_X$. Then $(g_1 \star g_2)x = g_1(g_2x)$ by the second property of a group action. Since $g_2x = x$, we have $(g_1 \star g_2)x = g_1x = x$. So $g_1g_2 \in G_X$.

Associativity: Associativity comes from the group G.

Identity: The identity $e \in G$ acts trivially on X by the first group action property, so $ex = x$ for all $x \in X$. Therefore $e \in G_X$.

Inverse: Let $g \in G_X$ and $x \in X$. Then:

$$\begin{aligned} x &= ex \quad \text{(by property 1 of group actions)} \\ &= (g^{-1} \star g)x \quad \text{(by the definition of inverses)} \\ &= g^{-1}(gx) \quad \text{(by property 2 of group actions)} \\ &= g^{-1}x \quad \text{(since } g \in G_X\text{)} \end{aligned}$$

So $g^{-1}x = x$ for all $x \in X$.

Therefore $g^{-1} \in G_X$. Therefore G_X is a subgroup of G.

G_X is normal in G: Consider $g \in G_X$ and $a \in G$. We want to prove that $aga^{-1} \in G_X$. Let $x \in X$. Then

$$\begin{aligned} (aga^{-1})x &= a(g(a^{-1}x)) \quad \text{(by property 2 of group actions)} \\ &= a(a^{-1}x) \quad \text{(since } a^{-1}x \in X \text{ and } g \text{ fixes all members of } X\text{)} \\ &= (aa^{-1})x \quad \text{(by property 2 of group actions)} \\ &= ex \quad \text{(by the definition of inverse)} \\ &= x \quad \text{(by property 1 of group actions)} \end{aligned}$$

So $aga^{-1} \in G_X$ for all $a \in G$. Therefore G_X is a normal subgroup of G.

Problem 3. (21 pts) Let $G = \mathbb{Z}_8 \times \mathbb{Z}_9$ and let $H = \langle(4,6)\rangle$.

(a) What is the index $(G : H)$ of H in G?

SOLUTION: We know that $|G| = 8{\cdot}9 = 72$ and $H = \{(4,6),(0,3),(4,0),(0,6),(4,3),(0,0)\}$, so $|H| = 6$. Since G is finite, $(G : H) = \frac{|G|}{|H|} = \frac{72}{6} = 12$.

(b) The group G/H is isomorphic to which product of cyclic groups?

SOLUTION: The fundamental theorem of finitely generated abelian groups implies that the only abelian groups of order 12 are $\mathbb{Z}_2 \times \mathbb{Z}_6$ and $\mathbb{Z}_{12}$. The group $\mathbb{Z}_{12}$ is cyclic. Consider the element $(1,1)H$ of G/H and the subgroup it generates:

$\langle(1,1)H\rangle = \{(1,1)H, (2,2)H, (3,3)H, (4,4)H, (5,5)H, (6,6)H, (7,7)H, (0,8)H, (1,0)H, (2,1)H, (3,2)H, (4,3)H = H\}$. Then $|\langle(1,1)H\rangle| = 12$, so $\langle(1,1)H\rangle = G/H$. Therefore G/H is cyclic and hence isomorphic to $\mathbb{Z}_{12}$.

(c) What is the order of $(2,2)H$ in G/H?

SOLUTION: The order of $(2,2)H$ in G/H is the smallest k such that $(2k, 2k) \in H$. Now consider the subgroup generated by $(2,2)H$:

$\langle(2,2)H\rangle = \{(2,2)H, (4,4)H, (6,6)H, (0,8)H, (2,1)H, (4,3)H = H\}$. Therefore the order of $(2,2)H$ is 6 in G/H.

Problem 4. (25 pts) Mark each of the following true or false. You do not need to justify your answers.

T (F) (a) The subgroup $\langle(1,2)\rangle$ is a normal subgroup of S_3.

SOLUTION: $(1,3)(1,2)(1,3) = (2,3)$ but $(2,3) \notin \langle(1,2)\rangle$.

T (F) (c) Any two abelian groups of order 24 are isomorphic.

SOLUTION: $\mathbb{Z}_6 \times \mathbb{Z}_4$ is not isomorphic to $\mathbb{Z}_{24}$ since $\gcd(6,4) \neq 1$.

(T) F (b) Any two groups of order 17 are isomorphic.

SOLUTION: If $|G| = 17$, then if $a \neq e$, $\langle a\rangle \neq \{e\}$. But Lagrange's Theorem implies that $|\langle a\rangle|$ divides 17. The only positive integer greater than 1 which divides 17 is 17, so G is a cyclic group of order 17.

(T) F (d) The set $\mathbb{Z}_n$ is a ring under addition and multiplication modulo n.

SOLUTION: We know it's a group under addition mod n, and it's closed under multiplication mod n. The distributive laws are true for integers and hence are true modulo n as well.

T (F) (e) The set $U_n = \{z \in \mathbb{C} | z^n = 1\}$ is a ring under ordinary addition and multiplication.

SOLUTION: U_n is not closed under addition, so it is not a ring.

Problem 5. (10 pts) Use the Sylow theorems to prove that no finite group of order 28 is simple.

SOLUTION: The third Sylow theorem implies that the number of Sylow 7-subgroups of G is congruent to $1 \mod 7$ and divides 28. The possibilities are $1, 8, 15$, and 22, but only 1 divides 28. Therefore there is only one Sylow 7-subgroup of G.

The second Sylow theorem implies that if H is a Sylow 7-subgroup of G, then gHg^{-1} is a Sylow 7-subgroup of G for all $g \in G$. Since there is only one such subgroup, $gHg^{-1} = H$ and hence H is normal.

Therefore G contains a proper non-trivial normal subgroup, so G is not simple.

Problem 6. (15 pts) Let G be an arbitrary group and let M and N be normal subgroups of G. Recall $NM = \{nm \mid n \in N, m \in M\}$. Prove that

$$\frac{NM}{M} \cong \frac{N}{N \cap M}.$$

SOLUTION:

Let $\phi : N \to \frac{NM}{M}$ be the map which sends n to the coset neM, where e is the identity in G. We know that $ne \in NM$, since $M \leq G \Rightarrow e \in M$.

ϕ is a homomorphism:

$$\begin{aligned}\phi(n_1 n_2) &= n_1 n_2 eM \\ &= n_1 e n_2 eM \\ &= (n_1 e)(n_2 e)M \\ &= (n_1 eM)(n_2 eM) \\ &= \phi(n_1)\phi(n_2)\end{aligned}$$

$\ker(\phi) = N \cap M$:

$$\begin{aligned}\ker(\phi) &= \{n \in N | \phi(n) = M\} \\ &= \{n \in N | neM = M\},\end{aligned}$$

since M is the identity in $\frac{NM}{M}$. We saw in class that $neM = M \iff ne \in M$. We know that $en = ne = n$ by the definition of the identity element e. So $neM = M \iff n \in M$. So the kernel of ϕ is the set of all elements of N which are also in M. Therefore $\ker(\phi) = N \cap M$.

The Fundamental Homomorphism Theorem states that $\frac{N}{\ker \phi} = \frac{N}{N \cap M}$ is isomorphic to $\phi[N]$. But the elements of $\frac{NM}{M}$ are the cosets $nmM = nM$ for $n \in N, m \in M$. Given $nM \in \frac{NM}{M}$, we see that $nM = \phi(n)$. So the image of ϕ equals $\frac{NM}{M}$, and hence $\frac{NM}{M} \cong \frac{N}{N \cap M}$.