ON EXTREMIZERS FOR ADJOINT FOURIER RESTRICTION INEQUALITIES AND A RESULT IN INCIDENCE GEOMETRY, ABSTRACT

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Whenever we have a bounded linear operator $T : X \to Y$ between two Banach spaces X, Y we can ask what nonzero elements $x^* \in X$ satisfy $\|Tx^*\| = \|T\|\|x^*\|$. Such elements of X are called extremizers for the inequality $\|Tx\| \leqslant \|T\|\|x\|$. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X satisfying $\|x_n\| \leqslant 1$ and $\|Tx_n\| \to \|T\|$, as $n \to \infty$, is called an extremizing sequence. For extremizing sequences we can ask whether they are precompact after the application of symmetries of the operator T. We can also ask for the value of the operator norm of T, $\|T\|$.

The adjoint Fourier restriction operator associated to a hypersurface S with measure σ in $\mathbb{R}^d$, $f \mapsto \widehat{f\sigma}$, is known to be bounded from L^2 to L^p in the case of the cone, the hyperboloid and the paraboloid in $\mathbb{R}^d$, for a certain range of exponents $p \in [1, \infty]$. Existence and nonexistence of extremizers, precompactness of extremizing sequences, Euler-Lagrange equations for extremizers and best constants is what we study in the first three parts of this dissertation.

In the first part we study the adjoint restriction inequality on the cone, $\Gamma^2 \subset \mathbb{R}^3$. We prove that extremizing sequences for the inequality from $L^2(\Gamma^2)$ to $L^6(\mathbb{R}^3)$ are precompact up to the natural symmetries of the cone, dilations and Lorentz transformations.

In the second part we study extremizers on the hyperboloid in dimensions 3 and 4. We prove that in both cases extremizers do not exist and compute the best constant in the adjoint Fourier restriction inequality.

In the third part, in a joint work with Michael Christ, we consider the case of the paraboloid, or equivalently, Strichartz inequalities for the Shrödinger equation. It is shown there that a natural class of functions, the Gaussians, known to extremize the $L^2 \to L^p$ adjoint Fourier restriction inequalities in dimensions 2 and 3 are no longer critical points, and thus are not extremizers, of the nonlinear functional associated to the $L^q \to L^p$ inequalities for $q \neq 2$. The case of mixed norms is also studied.

In the last chapter we look at an incidence geometry problem, the problem of counting noncoplanar intersections of lines in $\mathbb{R}^d$. The problem can be seen as a discrete version of the Kakeya problem, an open problem in real analysis. There we prove a sharp upper bound for the number of transverse intersections of a collection of lines.

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