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The least you need to know about differential equations

• Actually not only differential equations, do **not** neglect integration and power series!
• What is a differential equation?
• What is difference between general and particular solution?
• Direction fields for DE of form $y' = F(x, y)$
• Euler's method
• separation of variables, recognizing when to use it
• exponential growth and decay,

$$\frac{dP}{dt} = kP, \qquad P(0) = P_0$$

has solution $P(t) = P_0 \cdot e^{kt}$.
• logistic equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right), \qquad P(0) = P_0$$

has solution

$$P(t) = \frac{K}{1 + Ae^{-kt}}, \qquad A = \frac{K - P_0}{P_0}.$$

solve by separation of variables, using partial fractions.
• Behaviour of solutions to any differential equation when $t \to \infty$
• linear differential equation $y' + P(x)y = Q(x)$.

- integration factor.
- If $Q = 0$ (homogeneous), so $y' + P(x)y = 0$, try $y = e^f$ for some function f.
- initial value problems.

• second order linear differential equations,

$$P(x)y'' + Q(x)y' + R(x)y = S(x).$$

- principal of superposition: y_1, y_2 solution for $S = G_1$ and $S = G_2$ respectively, then $Ay_1 + By_2$ is solution for $S = AG_1 + BG_2$.
- the equation is homogeneous if $S(x) = 0$.
- auxiliary or complementary equation is the same equation with 0 instead of $S(x)$.
- general solution of homogeneous second order linear differential equation is given by $Ay_1 + By_2$ where y_1 and y_2 are linearly independent solutions.
- general solution of inhomogeneous second order linear differential equation is given by $f + Ay_1 + By_2$ where f is any solution and y_1 and y_2 are linearly independent solutions of the auxiliary equation.

- boundary problems, initial value problems

• second order constant linear differential equations, $ay'' + by' + cy = 0$.

- if $D = b^2 - 4ac > 0$, then solutions $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$ with $r_1, r_2 = (-b \pm \sqrt{D})/2a$.
- if $D = b^2 - 4ac = 0$, then solutions $y_1 = e^{rx}$ and $y_2 = xe^{rx}$ with $r = -b/2a$.
- if $D = b^2 - 4ac < 0$, then solutions $y_1 = e^{\alpha x}\cos(\beta x)$ and $y_2 = e^{\alpha x}\sin(\beta x)$, with $\alpha = -b/2a$ and $\beta = \sqrt{-D}/2a$, i.e., $\alpha \pm i\beta$ are solution to $ar^2 + br + c = 0$.

• undetermined coefficients. To find solution to inhomogeneous differential equations of the form $ay'' + by' + cy = G(x)$, try the following

$G(x)$	first guess	f.g. is solution to aux.eqn.
polynomial degree d	polynomial degree d	polynomial degree $d + 1$.
e^{kx}	Ae^{kx}	Bxe^{kx}
$\sin(kx)$	$A\sin(kx) + B\cos(kx)$	$x(A\sin(kx) + B\cos(kx))$
$(a_2x^2 + a_1x + a_0)e^{kx}$	$(b_2x^2 + b_1x + b_0)e^{kx}$	$(c_3x^3 + c_2x^2 + c_1x + c_0)e^{kx}$

• variation of parameters. For differential equation $ay'' + by' + cy = G(x)$ with y_1 and y_2 linearly independent solutions to the auxiliary equation, try $y = u_1y_1 + u_2y_2$ as a solution for the inhomogeneous equation. u_1 and u_2 give a solution if they satisfy

$$u_1'y_1 + u_2'y_2 = 0$$
$$u_1'y_1' + u_2'y_2' = G(x)a^{-1}$$

Solutions are given by

$$u_1' = \frac{y_2G(x)}{a(y_1'y_2 - y_2'y_1)}, \qquad u_2' = \frac{y_1G(x)}{a(y_2'y_1 - y_1'y_2)},$$

which give u_1 and u_2 by integration.
• using power series to find solutions to differential equations.