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Upper bounds for (quotients of) functions

Sometimes we want to find an upper bound of a function on some interval. For instance to get an upper bound for errors made in approximating integrals. This interval could consist of all real numbers, but could also be an interval of the form $[a, b]$. Examples are

function	interval	upper bound
$\sin x$	$\mathbb{R}$	1
$\sin x$	$\mathbb{R}$	23.6
$1/x$	$[2, 10]$	$1/2$
$4 - x^2$	$[-4, 4]$	4
$\frac{1}{1+x^2}$	$\mathbb{R}$	1

Some tricks are useful to find upper bounds of complicated functions. Try to break the function up in parts and use some of the following rules. Also make sure you know why the rules are true.

- $\sin x \leq 1$.
- If a function f is decreasing on an interval $[a, b]$, then the value $f(a)$ is an upper bound for f on that interval. (you need to know that it is decreasing though).
- If a function f is increasing on an interval $[a, b]$, then the value $f(b)$ is an upper bound for f on that interval.
- Suppose a function f is the sum of two other functions, say $f = g + h$. **If** L is an upper bound for g and M is an upper bound for h, then $L + M$ is an upper bound for f (all on the same interval).
- Suppose a function f is the product of two other functions, say $f = gh$. **If** L is a **positive** upper bound for g and M is a **positive** upper bound for h, then LM is an upper bound for f (all on the same interval).
- **watch out here** Suppose a function f is the quotient of two other functions, say$f = \frac{g}{h}$. **If** L is a **positive upper bound** for g and M is a **positive lower bound** for h, then $\frac{L}{M}$ is an upper bound for f (all on the same interval).
- If you have to give an upper bound for absolute values of functions, use the fact that $|f(x) + g(x)| \leq |f(x)| + |g(x)|$ for all x.
- Use facts such as $x^2 \geq 0$, so $1 + x^2 \geq 1$, so $\frac{1}{1+x^2} \leq 1$. Many similar things are possible.

Two examples. Give an upper bound for the function

$$\left| \frac{(\sin x + \cos x)(3x + 4)}{(1 + (x - 1)^4)^3} \right|$$

on the interval $[0, 4\pi]$. First of all, we note that this is the quotient of two functions, namely $|(\sin x + \cos x)(3x + 4)|$ and $(1 + (x - 1)^4)^3$. The first one is itself a product of functions, namely $|\cos x + \sin x|$ and $|3x + 4|$.

Now we are going to use several of the rules above. Since $\sin x \leq 1$ and $\cos x \leq 1$, we get $|\sin x + \cos x| \leq |\sin x| + |\cos x| \leq 1 + 1 = 2$, so 2 is an upper bound for $|\sin x + \cos x|$ on any interval. Clearly, since $|3x + 4|$ is increasing, an upper bound on the interval $[0, 4\pi]$ is $3 \cdot 4\pi + 4 = 12\pi + 4$. By the product rule above we find that $2 \cdot (12\pi + 4)$ is an upper bound for the product $|(\sin x + \cos x)(3x + 4)|$.

Since $(1 + (x - 1)^4)^3$ is in the denominator, we need to find a lower bound for it. Clearly, $(x - 1)^4$ is nonnegative, so $1 + (x - 1)^4 \geq 1$ and $(1 + (x - 1)^4)^3 \geq 1$, so 1 is a lower bound.

Using the upper bound $24\pi+8$ for the numerator $|(\sin x+\cos x)(3x+4)|$ and the lower bound 1 for the denominator $(1+(x-1)^4)^3$, we find by the quotient rule above that an upper bound for the entire function is given by $(24\pi+8)/1 = 24\pi+8$.

The second example comes from the Quiz 3. We want an upper bound for the function

$$f(x) = \left|\frac{-2x}{(1+x^2)^2}\right|$$

on the interval $[1, 10]$. One way is to do this similar as above, namely by realizing that this is a quotient. Clearly 20 is an upper bound for the numerator and similar as above, we find that 1 is a lower bound for the denominator, so an upper bound for the entire thing f is given by $20/1 = 20$.

Another way of finding an upper bound is using the fact that f is decreasing. That means that $f(1) = \frac{2}{2^2}$ is an upper bound of f. However, we do need to show that f is indeed decreasing. That can be done by taking its derivative and showing that $f'(x) \leq 0$ on the interval $[1, 10]$.

To prove that f is decreasing however, it is not enough to say that the numerator is just linear and the denominator grows faster, because it is of degree 4. To see that this is not enough, look at the function

$$g(x) = \frac{x}{(1+\frac{1}{100}x^2)^2}.$$

Again, it is **tempting** to say that "the numerator is just linear and the denominator is of degree 4, so the denominator grows faster than the numerator, and therefore the function is decreasing, so an upper bound for g on the interval $[1, 1000]$ is given by $g(1)$". This is **not true**. For instance we have $g(2) > g(1)$ (check yourself), so $g(1)$ is not an upper bound on the entire interval. This shows we have to be careful and precise and prove that a function is decreasing if we want to use it.

The other method however, may give only a rough estimate, but if we only need some upper bound, it can be much faster.